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MISCELLANEOUS

HISTORY, BIOGRAPHY

- ★ Fréchet, Maurice. *Les mathématiques et le concret*. Presses Universitaires de France, Paris, 1955. viii+438 pp. 1500 francs.
This is a collection of a number of papers by Fréchet grouped under the general headings: I) Sur les mathématiques en général. II) Sur le calcul des probabilités et ses applications. III) Les mathématiciens et la vie. Deux exemples. Papers have been selected for their general interest and readability by non-mathematicians.
- Kurepa, G. *Le rôle des mathématiques et du mathématicien à l'époque contemporaine*. Enseignement Math. (2) 1 (1955), 93-111.
- Kamke, E. *Die Rolle der Mathematik im heutigen Leben*. Enseignement Math. (2) 1 (1955), 112-134. (French summary)
- Weinberger, Otto. *Über die Anwendung der Mathematik auf Staatswissenschaften*. Enseignement Math. (2) 1 (1955), 135-148. (French summary)
- Darmois, Georges. *Rôle du mathématicien dans la vie contemporaine*. Enseignement Math. (2) 1 (1955), 149-158.
- van Dantzig, D. *The function of mathematics in modern society and its consequence for the teaching of mathematics*. Enseignement Math. (2) 1 (1955), 159-178. (French summary)
- Ascoli, Guido. *La funzione della matematica e del matematico nella vita contemporanea*. Enseignement Math. (2) 1 (1955), 179-187. (French summary)
- Gleason, A. N. *The expanding role of mathematics*. Enseignement Math. (2) 1 (1955), 188-191. (French summary)
- ★ Hauser, G. *Geometrie der Griechen von Thales bis Euklid, mit einem einleitenden Abschnitt über die vorgriechische Geometrie*. Eugen Haag, Luzern, 1955. 176 pp.
- Frenkian, Aram. *Etudes de mathématiques suméro-akkadiennes, égyptiennes et grecques*. Rev. Univ. "C. I. Parhon" Politehn. București. Ser. Ști. Nat. 2 (1953), no. 3, 9-20. (Romanian. Russian and French summaries)
- Raik, A. E. *From the early history of algebra. Quadratic equations among the Babylonians*. Molotov. Gos. Univ. Uč. Zap. 8, no. 1 (1953), 31-63. (Russian)
- van der Waerden, B. L. *Les mathématiques appliquées dans l'antiquité*. Enseignement Math. (2) 1 (1955), 44-45.
- Kennedy, E. S., and Transue, W. R. *A medieval iterative algorism*. Amer. Math. Monthly 63 (1956), 80-83.
- ★ Yuškevič, A. P. *On attainments of Chinese scholars in the field of mathematics*. Iz istorii nauki i tehniki Kitaya [From the history of science and engineering in China], pp. 130-159. Izdat. Akad. Nauk SSSR, Moscow, 1955. 7.75 rubles.
Essentially the same as a paper by the author in Istor.-Mat. Issled. 8 (1955), 539-572; MR 17, 1.
- Poincaré, Henri. *L'état actuel et l'avenir de la physique mathématique*. Gaz. Mat., Lisboa 15 (1955), no. 60-61, 3-16.
Reprinted from the original in Bull. Sci. Math. (2) 28 (1904), 302-324.
- Bergmann, Peter G. *Fifty years of relativity*. Science 123 (1956), 487-494.
- Ryago, G. *From the life and activity of four remarkable mathematicians of the University of Tartu*. Tartu. Gos. Univ. Trudy Estest.-Mat. Fak. 37 (1955), 74-105. (Russian. Estonian summary)
The mathematicians in question are J. M. C. Bartels, F. Minding, F. E. Molien and G. V. Kolosov.
- ★ Gordevskii, D. Z. *K. A. Andreev-vydayuščisya russkii geometr*. [K. A. Andreev - an outstanding Russian geometer.] Izdat. Har'kov. Gosudarstv. Univ., Kharkov, 1955. 46 pp. (1 plate). 1.30 rubles.
- ★ Ahiezer, N. I. *Akademik S. N. Bernšteĭn i ego raboty po konstruktivnoi teorii funktsii*. [Academician S. N. Bernšteĭn and his work on the constructive theory of functions.] Izdat. Har'kov. Gosudarstv. Univ., Kharkov, 1955. 112 pp. (1 plate). 3 rubles.
Somewhat amplified version of a paper in Uspehi. Mat. Nauk (N.S.) 6 (1951), no. 1(41), 3-67 [MR 12, 808].
- Montel, Paul. *Notice nécrologique sur Emile Borel*. C. R. Acad. Sci. Paris 242 (1956), 845-850.
- ★ Cartan, Elie. *Oeuvres complètes. Partie III. Vol. 1. Divers, géométrie différentielle. Vol. 2. Géométrie différentielle (suite)*. Gauthier-Villars, Paris, 1955. Vol. 1: xii+pp. 1-920 (1 plate); vol. 2: viii+921-1875. For part I and a general description of these collected works see MR 14, 343; for part II see MR 15, 383.

Have ★ Denjoy, Arnaud. Articles et mémoires. I. La variable complexe. Gauthier-Villars, Paris, 1955. x+pp. 1-507.

★ Denjoy, Arnaud. Articles et mémoires. II. Le champ réel - Notices. Gauthier-Villars, Paris, 1955. vi+pp. 509-1108.

These two volumes reproduce by photo-offset printing 14 selected papers of the author's on functions of a complex variable (vol. I), 15 selected papers on various aspects of the theory of real variables, a paper by T. J. Boks, and the author's own "Notices" concerning his work (vol. II.).

von Laue, M. Einstein und die Relativitätstheorie. Naturwissenschaften 43 (1956), 1-8.

Potapov, V. S. The work of V. P. Ermakov on vector algebra. From the history of mathematics. Stalingrad. Gos. Ped. Inst. Uč. Zap. 1953, no. 3, 3-8. (Russian)

Henri Fehr, 1870-1954. Sa vie et son oeuvre. Enseignement Math. (2) 1 (1955), 5-17 (1 plate).

Have ★ Fourier, Joseph. Analytical theory of heat. Translated, with notes, by Alexander Freeman. Dover Publications, Inc., New York, 1955. xxiii+466 pp. \$1.95.
Reprint by photo-offset of the first edition [Cambridge, 1878].

★ Bagratuni, G. V. Karl Fridrih Gauss. Kratkii očerk geodezičeskikh issledovanii. [Karl Friedrich Gauss. A short sketch of his investigations in geodesy.] Izdat. Geodezičeskoi Literatury, Moscow, 1955. 43 pp. 1 ruble.

Otradnyh, F. P. On the 250th anniversary of L. Magnitskii's "Arithmetic". Vestnik Leningrad. Univ. 1953, no. 11, 67-71. (Russian)

Rozenfeld, B. A. On the mathematical works of Muhammad Nasireddin. Izv. Akad. Nauk Azerbaidžan. SSR. 1953, no. 4, 35-50. (Azerbaijani)
Azerbaijani version of an earlier Russian paper [Istor.-Mat. Issled. 4 (1951), 489-512; MR 14, 524].

Cassina, Ugo. Storia ed analisi del "Formulario completo" di Peano. II, III. Boll. Un. Mat. Ital. (3) 10 (1955), 544-574.

Žautykov, O. K. P. Persidskii. (On his 50th birthday.) Vestnik Akad. Nauk Kazan. SSR. 1953, no. 11 (104), 46-50. (Russian)

Reymond, Arnold. A la mémoire de Pierre Sergescu (1893-1954). Enseignement Math. (2) 1 (1955), 21-29.

Agostinelli, Cataldo. Carlo Somigliana e la sua opera scientifica. Univ. e Politec. Torino. Rend. Sem. Mat. 14 (1954-55), 5-30 (1 plate).

A list of Somigliana's (1860-1955) published work is included.

Have ★ Stevin, Simon. The principal works of Simon Stevin. Vol. I. General introduction. Mechanics. Edited by E. J. Dijksterhuis. Swets & Zeitlinger, Amsterdam, 1955. v+617 pp. (1 plate). 52 guilders.

Following a general introduction on Stevin's life and work, his works on mechanics are reproduced by photo-offset from the original Dutch editions, with English translations on the facing pages.

★ Naumov, I. A. Dmitrii Matveevič Sincov. (Očerk žizni i naučno-pedagogičeskoi deyatelnosti.) [Dmitrii Matveevič Sincov. (A sketch of his life and scientific and pedagogical activity.)] Izdat. Har'kov. Gosudarstv. Univ., Kharkov, 1955. 72 pp. (1 plate). 2 rubles.

See also: van Rootselaar, p. 722.

FOUNDATIONS, MATHEMATICAL LOGIC

Ladrière, Jean. Mathématiques et formalisme. Rev. Questions Sci. (5) 16 (1955), 538-574.

Starting from the distinction between mathematics and metamathematics, the author describes two opposite tendencies in mathematical thought: the formal and the problematic (or realistic) tendency. The concrete development of mathematics implies a dialectics of the concept. Finally, mathematical experience is integrated into a metaphysical perspective. These rather speculative discussions are, in the mind of the author, certainly connected with concrete topics in the field of foundations but, unfortunately, these connections are not made sufficiently explicit. E. W. Beth (Amsterdam).

Have ★ Heyting, A. Intuitionism. An introduction. North-Holland Publishing Co., Amsterdam, 1956. viii+133 pp. 13.50 guilders.

This is an introduction to intuitionistic mathematics for mature mathematicians. The reader is taken rapidly to the heart of several different branches of intuitionistic mathematics. The speed of development is achieved by condensing the proofs and by presuming familiarity with the classical counterparts to the theories discussed.

The book is written as a dialogue between Class (a

classical mathematician), Form (a formalist), Int (an intuitionistic mathematician), Letter (a finitistic nominalist), Prag (a pragmatist), and Sign (a signficist). In the first chapter Int defends intuitionistic mathematics against the criticism of the others, asking them finally to judge for themselves. In the remaining chapters Int presents mathematics for them to judge. In these chapters Class, except for Int, is the most loquacious; he frequently compares classical results with corresponding intuitionistic results and his questions lead Int to a more detailed discussion of some points. The device of dialogue allows abbreviation of statements without loss of clarity.

Chapter 2, after a rapid introduction of natural numbers, integers and rationals, is devoted to real number-generators (convergent sequences of rationals) including a brief discussion of limits of sequences of real number-generators. Real numbers can be introduced in Chapter 3 after the introduction of spreads and species. The remainder of this chapter is devoted to a proof of the fundamental theorem for finitary spreads and a discussion of its consequences. Chapter 4 is devoted mainly to linear equations, the reader being referred for further information on intuitionistic algebra to an earlier paper [Verh. Nederl. Akad. Wetensch. Afd. Natuurk. Sect. I.

18 (1941), no. 2; MR 7, 405]. In chapter 5 sufficient theory of plane pointspecies (plane point sets) is presented for the development of the theory of measure and integration in chapter 6. These two chapters form one half (by page count) of the mathematics developed in the book. In chapter 7 the axioms for the intuitionistic propositional logic are justified by references to previous chapters. Some important theorems of the predicate calculus are listed and counter-examples are given for some classical theorems which are not intuitionistically valid. Finally the logic is applied to show how to one classical concept there may correspond several intuitionistic concepts. The final chapter of the book is entitled "Controversial Subjects" and discusses a method of definition by L. E. J. Brouwer [Nederl. Akad. Wetensch., Proc. 51 (1948), 963-964; Library of the Tenth International Congress of Philosophy, Amsterdam, 1948, Vol. I, Proceedings of the Congress, pp. 1235-1249 (1949); MR 10, 421, 422] as well as negationless intuitionistic mathematics.

Int is to be praised for a clear non-polemical presentation of the subject matter. *P. C. Gilmore.*

Valpola, Veli. *Ein System der negationslosen Logik mit ausschliesslich realisierbaren Prädikaten.* Dissertation, Universität Helsinki, 1955. Acta Philos. Fenn. 9 (1955), 247 pp.

The author concludes from a lengthy philosophical discussion that negation as well as the empty set ought to be banished from mathematics; his point of view resembles that of Griss [Nederl. Akad. Wetensch. Proc. Ser. A. 54 (1951), 41-49; MR 13, 97]. He then begins the construction of a corresponding system of symbolic logic. This was also done by Griss [ibid. 53 (1950), 456-463; MR 12, 3], Gilmore [ibid. 56 (1953), 162-174, 175-186; MR 14, 1053] and Vredenduin [Compositio Math. 11 (1953), 204-270; MR 15, 846]. Though the author mentions Griss' and Gilmore's papers, his work seems to be independent of theirs. His method resembles that of Vredenduin. Just as Griss and Vredenduin, he develops no calculus of propositions, but begins at once with a predicate calculus. However, he uses only one sort of variables, ranging over individuals, predicates and classes without type distinction. His treatment is detailed and limited to first principles; in particular, he does not treat the relation of distinctness. At the end of the paper he states without proof some metamathematical results about his system.

A. Heyting (Amsterdam).

Loś, J.; Mostowski, A.; and Rasiowa, H. *A proof of Herbrand's theorem.* J. Math. Pures Appl. (9) 35 (1956), 19-24.

With each sentence Y of a first-order theory T with open axioms there is associated a sequence $\{H_n\}$ of sentences (H_n is the n th "Herbrand alternation") in a definite combinatorial manner; the theorem is that Y is provable in T if and only if H_n is provable in T for some n . The proof is algebraic to the extent that it depends on the consideration of a "Lindenbaum algebra" (i.e., the lattice of equivalence classes of a suitable auxiliary theory, two sentences being equivalent if their biconditional is provable). The paper concludes with some remarks on possible extensions of the method that yield the Gödel completeness theorem in full generality, the Skolem-Löwenheim theorem, and related results for modal and intuitionistic logics.

P. R. Halmos (Chicago, Ill.).

★ **Skolem, Th.** *Peano's axioms and models of arithmetic.* Mathematical interpretation of formal systems, pp. 1-14. North-Holland Publishing Co., Amsterdam, 1955. \$3.25.

The first half of this paper is devoted to a proof that any set of first-order sentences which are true for the natural numbers has a denumerable model which is not the set of natural numbers. The demonstration differs only slightly from the author's previous proof [Fund. Math. 23 (1934), 150-161, pp. 157-159]. The rest of the paper contains an account of special models for certain fragments of number theory. (The assertion at the bottom of p. 12 seems to be in error; for example, $f(t) = 3t + 3/2$ and $g(t) = 3t + 1$ provide a counterexample.) *E. Mendelson.*

★ **Hasenjaeger, G.** *On definability and derivability.* Mathematical interpretation of formal systems, pp. 15-25. North-Holland Publishing Co., Amsterdam, 1955. \$3.25.

Following Henkin, the author defines the class D of standard models to be those models for systems containing predicate variables in which the ranges for the predicate variables are extensionally maximal. A wff B is valid in a model M if B is true for every substitution of constants from M for the free variables, of all types, occurring in B . Let K be a class of models. Then B is called a universal K -consequence of Γ (in symbols: $\Gamma \cup_K B$) if and only if, for every model M in K for which each wff in Γ is valid, B is valid also. If " $\Gamma \vdash B$ " means that B is derivable from Γ , adequacy of K for \vdash is expressed by: $\Gamma \vdash B$ if and only if $\Gamma \cup_K B$, for all Γ and B . The author proves that, for the pure first-order predicate calculus (with a substitution rule for predicates), the number-theoretic relation corresponding to the relation $\{A\} \cup_D B$ is not in $P_1^{(2)} \cup Q_1^{(2)}$. Hence there is no derivability relation for which D is adequate. Let \vdash_P represent derivability in the pure first-order calculus, and let K_0 be the class of models M such that, for any wff $A(x_1, \dots, x_n)$, and for any substitution on variables free in $\lambda x_1 \dots x_n A(x_1, \dots, x_n)$, the resulting relation belongs to the range in the model of n -place predicate variables. Then K_0 is adequate for \vdash_P , and, therefore, non-derivability proofs can be given by models in K_0 . A particular case of non-derivability is treated in detail by methods similar to those used by the author in J. Symb. Logic 15 (1950), 273-276 [MR 12, 578]. *E. Mendelson (Chicago, Ill.).*

★ **Kreisel, G.** *Models, translations and interpretations.* Mathematical interpretation of formal systems, pp. 26-50. North-Holland Publishing Co., Amsterdam, 1955. \$3.25.

This is a collection of notes and remarks, requiring, for minimum comprehension, a thorough knowledge of the literature. The main body of the paper treats of the mutual connections among the notions mentioned in the title, and their relation to consistency problems and informal mathematics. In an eleven page appendix, the author sketches a proof that Z is ω -consistent. The ω -consistency of Z is taken in the following form: If a formula

$$(Ey)(x_1)(Ey_1) \dots (x_n)(Ey_n)A(y; x_1, \dots, x_n; y_1, \dots, y_n)$$

can be proved in Z , there is an ordinal recursive functional $\Phi(\phi)$ of order $< \epsilon^2$ such that $\phi[\Phi(\phi)]$ is not (the number of a proof in Z of the formula

$$\neg(x_1)(Ey_1) \dots (x_n)(Ey_n)A(\Phi(\phi); x_1, \dots, x_n; y_1, \dots, y_n).$$

The author also indicates how a lemma of Gentzen may be used to show that the ω -consistency of arithmetic without induction cannot be proved in Z .
E. Mendelson.

Have ★ **Robinson, Abraham.** *Ordered structures and related concepts.* Mathematical interpretation of formal systems, pp. 51–56. North-Holland Publishing Co., Amsterdam, 1955. \$3.25.

Let K be a non-empty consistent set of statements of a first-order system involving certain relations and constants, and let M be a model of K . Add the elements of M as constants of K . The complete diagram N of M consists of all formulas $R(a_1, \dots, a_n)$ such that a_1, \dots, a_n are elements of M for which R holds, and all formulas $\neg R(a_1, \dots, a_n)$ such that a_1, \dots, a_n are elements of M for which R does not hold. The set K is called model-complete if, for every model M of K , the set $K \cup N$ is complete, N being the complete diagram of M . The author proves the following Main Theorem: A non-empty consistent set K is not model-complete if and only if there are models M and M' of K , and a statement X which contains only relations and constants of M , such that M' is an extension of M , and X is satisfied by M' although it is not satisfied by M . Moreover, X has the form $(\exists y_1) \dots (\exists y_k) F(y_1, \dots, y_k)$, where F is a conjunction of atomic statements and/or negations of such statements.

Of the two notions of completeness and model-completeness, neither implies the other. A model M of a set K of statements is called a prime model of K if every model of K is isomorphic to an extension of M . If a model-complete set K possesses a prime model, then K is complete. The author indicates several applications of his main theorem, in particular the results that the axioms for algebraically-closed fields, and those for real closed fields, are model-complete. In addition, Tarski's theorem that the axioms for real closed fields are complete can be deduced.
E. Mendelson (Chicago, Ill.).

Have ★ **Wang, Hao.** *On denumerable bases of formal systems.* Mathematical interpretation of formal systems, pp. 57–84. North-Holland Publishing Co., Amsterdam, 1955. \$3.25.

This is chiefly an expository discussion of an assortment of topics. Among the subjects treated are: (i) enumerability axioms; (ii) impossibility of a finite axiomatization of certain formal systems; (iii) systems of natural deduction using Hilbert's ϵ -operator; (iv) categoricity of a formal system relative to a subsystem or to a given set of predicates; (v) the Skolem paradox and non-standard models; (vi) completeness of quantification theory; (vii) the author's "constructive" system [J. Symb. Logic 19 (1954), 241–266; MR 16, 661], and the ordinals needed in the description of that system; and (viii) consistency proofs for certain impredicative theories.
E. Mendelson (Chicago, Ill.).

Have ★ **Henkin, L.** *The representation theorem for cylindrical algebras.* Mathematical interpretation of formal systems, pp. 85–97. North-Holland Publishing Co., Amsterdam, 1955. \$3.25.

If r is an ordinal and X a set, then $d_Y = \{x | x \in X^r, x_i = x_j\}$ and, for $Y \subseteq X^r$, the outer cylindricfication $C_r Y$ consists of all elements a of X^r which differ from some element b of Y at most in the i th term. An r -dimensional highly proper cylindrical algebra (h.p.c.a.) is a system $\langle A, \cup, \cap, \neg, d_Y, C \rangle$ such that, for some set X , A is a family of subsets of X^r containing all the elements d_Y , and closed under all

of the cylindricfications C_i as well as the Boolean operations of \cup , \cap , and \neg . The dimension DY of an element Y of an r -dimensional h.p.c.a. B is the set of ordinals $i < r$ such that $C_i Y \neq Y$. If DY is finite for every Y in B , B is said to be locally finite-dimensional (l.f.d.). Every l.f.d. ω -dimensional h.p.c.a. can be given as the sets of satisfying sequences in a model of a formal system.

Following Tarski and Thompson, to whom many of the ideas of this paper are credited, the author gives an axiom system for the notion of an r -dimensional abstract cylindrical algebra with diagonal elements (c.a.). In order to obtain a representation theorem for c.a.'s, the author generalizes the notion of an h.p.c.a. to that of a proper cylindrical algebra (p.c.a.), which is essentially a direct product of h.p.c.a.'s. Every p.c.a. is a c.a.; but there are other c.a.'s, for example, the Lindenbaum algebra of any formal system. An r -dimensional c.a. B is called dimension-complemented if the union of the dimension indices of any finite number of elements of B has an infinite complement with respect to r . The representation theorem, which is stated but not proved, says that any dimensionally-complemented r -dimensional c.a. is isomorphic to an r -dimensional p.c.a.. The only proof known to the author is metamathematical, and involves the notion of a formal system in which there are predicates with an infinite number of arguments. In a note, the author mentions several improvements of the representation theorem.
E. Mendelson (Chicago, Ill.).

Have ★ **Łoś, Jerzy.** *Quelques remarques, théorèmes et problèmes sur les classes définissables d'algèbres.* Mathematical interpretation of formal systems, pp. 98–113. North-Holland Publishing Co., Amsterdam, 1955. \$3.25.

An E -definable class of algebras consists of those algebras satisfying some set of formulas of a formal system. If this set can be chosen so as to consist only of free-variable formulas, then the class of algebras is said to be 0-definable. In connection with the theory of definable classes of algebras, the author indicates three directions of research.

The first direction has to do with characterizing certain types of definable classes of algebras by means of operations on the algebras. Birkhoff's theorem is a typical result in this direction, and the following results along the same lines are stated. (i) Using the notion of a logical field, the author gives a necessary and sufficient condition for a class of algebras to be 0-definable. (ii) Any 0-definable class is closed under the formation of sums of increasing sequences of algebras. An unpublished result of Ryll-Nardzewski about sums of E -definable algebras is stated and some applications mentioned. (iii) An affirmative answer to a hypothesis of Marczewski about classes closed with respect to division by congruence can be given. A characterization of classes closed with respect to summation is conjectured by the author. (iv) The following proposition posed by the author has been proved by R. L. Vaught: If, for each n , $\Gamma_1 \times \dots \times \Gamma_n$ belongs to an E -definable class A , then the infinite direct product $\prod_{n=1}^{\infty} \Gamma_n$ belongs to A .

The second direction of research treats of conditions under which certain classes of algebras defined in a non-elementary way are E -definable. For example, the author has shown [Bull. Acad. Polon. Sci. III. 2 (1954), 21–23; MR 16, 564] that the class of all groups which admit a linear order is 0-definable, but not arithmetic. More general results are stated, and mention is made of the

open problem as to whether the class P of all groups G in which every partial order can be extended to a linear order is E -definable.

In connection with the third class of problems, the author reports on some theorems which have been obtained on the existence of extensions of a given algebra and of common extensions of a class of algebras.

Theorem (1.3) probably should read: A system X is axiomatizable if and only if $Cn(X + \neg X) = E$. The given condition, $X \cdot \neg X = Cn(\emptyset)$, is true for all systems X .

E. Mendelson (Chicago, Ill.).

Faris, J. A. The Gergonne relations. *J. Symb. Logic* 20 (1955), 207-231.

Faris develops a formal system whose primitive elements are the second-degree proposition-forming functions '1', '2', '3', '5' and the term variables, 'a', 'b', 'c', 'd', 'd', 'd', 'd'. Expressions such as '2ab', '5ca' are called affirmative simple expressions. The only significant expressions of the system are those which can be formed by substituting affirmative simple expressions for the variables in any significant expression of the propositional calculus.

The system may be interpreted in terms of classes, 'a', 'b', etc., having for their range of values universal names, and the functions '1', '2', '3', '5' representing respectively the relations: "every a is a b and every b is an a ", "every a is a b and not every b is an a ", "not every a is a b and not every b is an a and some a is a b ", "no a is a b ".

Faris axiomatizes the system both for assertion and rejection and gives a table of asserted and rejected Gergonne syllogisms. He then points out that it contains an infinity of undecidable expressions and completes the system by means of a new rejection rule. *A. Rose.*

Matsumoto, Kazuo. Reduction theorem in Lewis' sentential calculi. *Math. Japon.* 3 (1955), 133-135.

Matsumoto makes, for the Lewis calculi S_4 and S_5 , the definition: $\Delta x \equiv \sim \Diamond \sim \Diamond \sim \alpha$ and shows that γ is provable in S_5 if, and only if, $\Delta \gamma$ is provable in S_4 . He then deduces that if γ is provable in S_5 , then $\Diamond \gamma$ is provable in S_4 . Finally he uses a theorem of Halldén [*J. Symb. Logic* 14 (1950), 230-236; MR 11, 303] to show that if γ is provable in S_4 then $\Diamond \Diamond \gamma$ is provable in S_3 .

A. Rose (Nottingham).

Nelson, Raymond J. Weak simplest normal truth functions. *J. Symb. Logic* 20 (1955), 232-234.

Nelson gives a calculation procedure for determining the class of simplest normal truth-functions equivalent to a given formula Φ under the hypothesis that certain conjunctions of letters of Φ are always false. The method is based on that of one of his previous papers [same *J.* 20 (1955), 105-108; MR 17, 224]. In conclusion Nelson raises, but does not discuss, the problem of determining necessary and sufficient conditions for a weak simplest truth-function to be simpler than a strong one. *A. Rose.*

Takeuti, Gaisi. On a generalized logic calculus. *Jap. J. Math.* 23 (1953), 39-96 (1954); errata, 24 (1954), 149-156.

The author presents some general ideas on the foundations of analysis, and a system (GLC) which is intended to back up these ideas. GLC is an elaborate formalization of the simple theory of types following Gentzen's investigations [*Math. Z.* 39 (1934), 176-210, 405-341]. The main result, which the author calls 'theorem of restriction',

states that a set of axioms A_1, \dots, A_N is consistent over GLC if, in the usual terminology, there is a model for A_1, \dots, A_N in a consistent extension of GLC; in the definition of 'model' the natural closure conditions are imposed on the models of objects of higher type. This implies, as is well-known, that Peano's arithmetic is consistent if GLC is consistent when extended by an axiom of infinity of the form: $(x)(y)(x' = y' \rightarrow x = y)$, $(Ex)(y)(x \neq y)(\Delta)$.

The author's general ideas are these: (i) a higher-order predicate calculus, i.e. the rules governing sets and functions, should be such that the addition of these concepts leave a consistent system consistent; (ii) more specifically, Gentzen's elimination of cuts (loc. cit.) should generalize to the higher-order predicate calculus. Condition (i) is satisfied by a strictly predicative higher-order predicate calculus, as shown by Novak [*Fund. Math.* 37 (1950), 87-110; MR 12, 791], but, in the reviewer's opinion, it is not satisfied by the author's system; for, every arithmetic proposition is expressed in G^1LC , i.e. that part of GLC which is restricted to special functions and variables for individuals and predicates of individuals; also GLC seems to be sufficiently impredicative to derive from Δ the (arithmetic) formula $Con_1 \Delta$ which expresses the consistency of Δ over G^1LC . So either Δ is ω -inconsistent over G^1LC or else $\Delta \& \neg Con_1 \Delta$ is consistent over G^1LC , but not over GLC, contradicting (i).

G. Kreisel (Princeton, N.J.).

Izumi, Yoshihisa. Sur le degré de la perfection. *Tôhoku Math. J.* (2) 7 (1955), 128-131.

Izumi claims to have disproved two theorems of the reviewer [*Math. Z.* 54 (1951), 181-183; *J. Symb. Logic* 16 (1951), 204; MR 13, 614; 309], but his argument appears to contain several errors. For example, it is claimed that the set of formulae satisfying the 3-valued tables:

$x \sim y$	y		$x \sim y$	y	
	0	1		0	1
0	0	1	0	0	1
1	1	0	1	0	1
2	2	2	2	2	1

(presumably with 0 as the only designated value) is identical with the set of tautologies constructed from \sim and \approx . In fact the tautology $(x \sim y) \sim ((z \sim x) \sim (z \sim y))$ takes the value 1 when x, y, z take the values 0, 1, 2 respectively. *A. Rose (Nottingham).*

Izumi, Yoshihisa; et Wada, Tôru. Sur la notion de la perfection. *Tôhoku Math. J.* (2) 7 (1955), 132-135.

The authors attempt to prove some theorems concerning the relationship between truth-tables and formalizations. The notation is inadequately explained and the paper is very difficult to follow. *A. Rose (Nottingham).*

Lacombe, Daniel. Remarques sur les opérateurs récursifs et sur les fonctions récursives d'une variable réelle. *C. R. Acad. Sci. Paris* 241 (1955), 1250-1252.

A correction to and some extensions of the work reported in three previous notes of the author [same *C. R.* 240 (1955), 2478-2480; 241 (1955), 13-14, 151-153; MR 17, 225] in which a development of the concept of a recursive function of a real variable was outlined. Among the topics considered in the present note are partially defined functions, uniform continuity, and the set of

maximal values of a function on a closed interval whose endpoints are recursive real numbers.

H. G. Rice (Durham, N.H.).

Behrend, F. A. A contribution to the theory of magnitudes and the foundations of analysis. *Math. Z.* 63 (1956). 345-362.

The additive semigroup P of positive real numbers ("magnitudes") is characterized by the following postulates: (i) trichotomy law (P is simply ordered); (ii) addition is associative; (iii) $a < b$ is equivalent to the solvability of $a + x = b$ and to that of $y + a = b$; (iv) Dedekind cut axiom. If (iv) is weakened to Archimedes' axiom, and (ii) and (iii) weakened to one-sided conditions, one gets postulates for the class Σ of additive sub-semigroups of P . One can construct the exponential and logarithmic functions by showing that, relative to any "unit" e (multiplicative identity), the $x > e$ form a multiplicative semigroup satisfying (i)-(iv) and hence isomorphic to P . Finally, "incomplete systems" (like positive angles) are characterized; following L. Rieger [*Věstník Královské České Společnosti Nauk. Třída Matemat.-Přirodověd.* 1946, no. 6; 1947, no. 1; 1948, no. 1; MR 9, 7; 10, 99], they are shown to be images of analogs of covering groups.

G. Birkhoff (Cambridge, Mass.).

Renaud, Paul; Joly, Maurice; et Dervichian, Dikran G. Notion de fréquence de présence d'une grandeur mesurable G . *C. R. Acad. Sci. Paris* 240 (1955), 2384-2387.

On considère les paramètres a_i exprimant les relations entre une grandeur G et l'appareil de mesure. Il existe un domaine D de ces paramètres où la mesure ne varie pas. La grandeur de D définit la fréquence de présence de G .

Si l'on postule que l'existence expérimentale d'une grandeur G est définie par la possibilité de déterminer une mesure G_m unique, avec une précision donnée, la fréquence de présence de G représente alors une évaluation de son existence expérimentale pour cette précision. On étend aux cas des phénomènes. On compare les fréquences de présence des phénomènes élémentaires et des phénomènes quelconques.

C. C. Torrance.

Destouches, Jean-Louis. Allgemeine Theorie der Voraussagen. *Arch. Math. Logik Grundlagenforsch.* 2 (1954), 10-14.

„Die allgemeine Theorie der Voraussagen wird gebildet von der Gesamtheit aller Überlegungen, welche sich beziehen auf die Berechnung der Voraussagen unabhängig von jeder Voraussetzung mit Ausnahme derjenigen, welche wir soeben angegeben haben.“ The author introduces a formal symbolism by which he outlines a solution of the problem of prediction.

C. C. Torrance.

Ottaviani, Giuseppe. Sul concetto di infinito nella matematica applicata. *Giorn. Ist. Ital. Attuari* 18 (1955), 59-70.

L'A. si occupa della differenza tra i concetti di numero comunque grande, di infinito come divenire e di infinito attuale, con riferimento ad alcune questioni del calcolo della probabilità e della matematica finanziaria.

Author's Summary.

★ **Smith, Vincent Edward.** St. Thomas on the object of geometry. *Marquette University Press, Milwaukee, Wis.*, 1954. vii+99 pp.

See also: Novikov, p. 706.

ALGEBRA

Schützenberger, Marcel Paul. Une théorie algébrique du codage. *C. R. Acad. Sci. Paris* 242 (1956), 862-864.

A code is defined as a correspondence between the elements of a set of "elementary messages" and certain sequences of letters, called "words", of a set of "letters". In this note the author indicates that the application of results from the theory of semi-groups due to P. Dubreil [*Mém. Acad. Sci. Inst. France* (2) 63 (1941), no. 3; MR 8, 15] leads to consequences bearing on the physical realization of coding and decoding machines.

S. Kullback.

Gould, H. W. Some generalizations of Vandermonde's convolution. *Amer. Math. Monthly* 63 (1956), 84-91.

Vandermonde's convolution is the relation obtained by equating coefficients of powers of x in the identity: $(1+x)^{q+r} = (1+x)^q(1+x)^r$, and of course relates binomial coefficients. The generalizations are for coefficients:

$$A_n(\alpha, \beta) = \binom{\alpha + \beta n}{n} \alpha(\alpha + \beta n)^{-1} \text{ and } B_n(\alpha, \beta) = \binom{\alpha + \beta n}{n}.$$

The author obtains "generating" functions of the type given by Lagrange's theorem as follows

$$A(x; \alpha, \beta) = \sum A_n(\alpha, \beta) z^n = x^\alpha,$$

$$B(x; \alpha, \beta) = \sum B_n(\alpha, \beta) z^n = x^{\alpha+1}[\beta + (1-\beta)x]^{-1},$$

where in both $z = (x-1)x^{-\beta}$. Convolutions follow from

$$A(x; \alpha + \gamma, \beta) = A(x; \alpha, \beta)A(x; \gamma, \beta),$$

$$B(x; \alpha + \gamma, \beta) = A(x; \alpha, \beta)B(x; \gamma, \beta).$$

These are used to verify old and new curious identities in binomial coefficients.

Reviewer's note: As the author remarks, the result above for $B(x; \alpha, \beta)$ appears as problem 216 of Pólya and Szegő, *Aufgaben und Lehrsätze* [Bd. I, III. Abschnitt, Springer, Berlin, 1925]. The result for $A(x; \alpha, \beta)$ follows at once from this and

$$A_n(\alpha, \beta) = B_n(\alpha, \beta) - \beta B_{n-1}(\alpha + \beta - 1, \beta).$$

J. Riordan (New York, N.Y.).

Mitropol'skii, A. K. On determinants of the distribution of a series of natural numbers. *Uspehi Mat. Nauk* (N.S.) 10 (1955), no. 4(66), 143-144. (Russian)

Let \bar{n} be the arithmetic average of the first n natural numbers; let S_i be $\sum_{j=1}^n |i - \bar{n}|^{2j}$. The author gives formulas for S_1, S_2, S_3, S_4 and does the necessary arithmetic for $n=1(1)25$. Similarly for D_2, D_3, D_4 , where D_k is the determinant of a certain symmetric matrix (continuant) with S_i for elements; viz.:

$$D_2 = \det \begin{pmatrix} S_0 & S_1 \\ S_1 & S_2 \end{pmatrix}; \quad D_3 = \det \begin{pmatrix} S_0 & S_1 & S_2 \\ S_1 & S_2 & S_3 \end{pmatrix};$$

$$D_4 = \det \begin{pmatrix} S_0 & S_1 & S_2 \\ S_1 & S_2 & S_3 \\ S_2 & S_3 & S_4 \end{pmatrix}.$$

J. L. Brenner (Pullman, Wash.).

Linear Algebra, Polynomials, Invariants

Krukovskii (Krukovskii-Sinevič), B. V. On a proposition of the theory of determinants and on its consequences. Kiev. Avtomob.-Dorož. Inst. Trudy. 1 (1953), 150-153. (Russian)

In a matrix, if two columns are equal, the determinant is zero. Similarly, if j columns are equal, except possibly in the first $n-j+2$ rows, the determinant is zero. If all the elements of any fixed j columns have a common factor, except possibly those elements in the first $n-j+1$ rows, the determinant is divisible by that common factor.

J. L. Brenner (Pullman, Wash.).

Hua, Loo-Keng. Inequalities involving determinants. Acta Math. Sinica 5 (1955), 463-470. (Chinese. English summary)

Let Z, W be complex-valued $n \times n$ matrices, Z', W' their conjugate transposes, and let I be the identity matrix. If $I - ZZ' > 0$, $I - WW' > 0$, then the following identity holds:

$$(1) \quad d(I - ZZ')d(I - WW') \leq |d(I - ZW')|^2,$$

where d denotes the determinant. More generally, let X_1, \dots, X_m be m square matrices of order n with complex elements, and $\rho > 0$. If $I - X_i X_i' > 0$, $1 \leq i \leq m$, then the Hermitian matrix

$$(2) \quad \begin{pmatrix} d(I - X_1 X_1')^{-\rho-n+1} & \dots & d(I - X_1 X_m')^{-\rho-n+1} \\ \vdots & \ddots & \vdots \\ d(I - X_m X_1')^{-\rho-n+1} & \dots & d(I - X_m X_m')^{-\rho-n+1} \end{pmatrix}$$

is positive semi-definite. The author gives an elementary proof of the first inequality. Proof of its generalization (2) makes use of the representation theory of linear groups. *S. Chern (Chicago, Ill.).*

Dionísio, J. Joaquim. A rule for computing the eigenvalues and the eigen-vectors of a permutation matrix. Rev. Fac. Ci. Univ. Coimbra 23 (1954), 53-55.

The author gives eigenvalues and eigenvectors of a general permutation matrix a by noting that a is the direct sum of m circulant matrices. *G. E. Forsythe.*

Parodi, Maurice. Sur une propriété des racines d'une équation qui intervient en mécanique. C. R. Acad. Sci. Paris 241 (1955), 1019-1021.

This note localizes the roots of the determinantal equation

$$(*) \quad \|a_{ij}z + b_{ij}\| = 0$$

of degree n , generalizing results for the case $a_{ij} = -\delta_{ij}$ due to Gerschgorin and others [see A. Ostrowski, Rend. Mat. e Appl. (5) 10 (1951), 156-168; MR 14, 125]. Following Ostrowski's methods, the author shows that each root of (*) lies in the union of the n ovals of Descartes

$$|z + b_{ii}a_{ii}^{-1}| - \lambda_{1i}|z| \leq \lambda_{2i} \quad (i=1, \dots, n)$$

for any positive p, q, k_1, \dots, k_n such that $p^{-1} + q^{-1} = 1$ and $\sum_{i=1}^n (k_i + 1)^{-1} \leq 1$, where

$$\lambda_{1i} = |a_{ii}|^{-1} k_i^{1/q} \left\{ \sum_{j \neq i} |a_{ij}|^p \right\}^{1/p},$$

$$\lambda_{2i} = |a_{ii}|^{-1} k_i^{1/q} \left\{ \sum_{j \neq i} |b_{ij}|^p \right\}^{1/p}.$$

Special results come for $q = \infty, p = 1$; and for $q = 1, p = \infty$. One can transpose (*) first and thus get column sums instead of row sums.

G. E. Forsythe (Los Angeles, Calif.).

Dionísio, J. Joaquim. The eigenvectors common to linear quasi-commutative operators. Gaz. Mat., Lisboa 15 (1955), no. 60-61, 22-24. (Portuguese)

The author gives a proof, claimed to be new, of the following theorem due to Drazin, Dungey, and Gruenberg [J. London Math. Soc. 26 (1951), 221-228; MR 12, 793]. Suppose each of the finite matrices A_1, \dots, A_m commutes with each matrix $A_i A_j - A_j A_i$ ($i, j = 1, \dots, m$). [Reviewer's note: The matrices A_1, \dots, A_m are thereby called "quasi-commutative," a term apparently previously applied only when $m=2$.] Let r_i be the number of distinct eigenvalues of A_i ; let $r = \max r_i$. Then the matrices A_i have at least r linearly independent eigenvectors in common. *G. E. Forsythe (Los Angeles, Calif.).*

Schneider, Hans. A matrix problem concerning projections. Proc. Edinburgh Math. Soc. (2) 10 (1956), 129-130.

If A is an $n \times m$ complex matrix of rank r , then there exists an $m \times n$ matrix B such that $(I - AB)^*(I - AB)$ is a projection of rank k , if and only if $k \geq n - r$. In fact, if $k \geq n - r$, B can be found such that $I - AB$ is a hermitian projection of rank k . This generalizes a result of H. Nagler [same Proc. (2) 10 (1953), 21-24; MR 14, 837].

N. G. de Bruijn (Amsterdam).

Sz.-Nagy, Béla. Remark on S. N. Roy's paper "A useful theorem in matrix theory". Proc. Amer. Math. Soc. 7 (1956), 1.

The remark is substantially that of the reviewer of Roy's paper [Duke Math. J. 21 (1954), 225-231; MR 16, 4]. *J. L. Brenner (Pullman, Wash.).*

Egerváry, E. On hypermatrices whose blocks are commutable in pairs and their application in lattice-dynamics. Magyar Tud. Akad. Alkalm. Mat. Int. Közl. 3 (1954), 31-47 (1955). (Hungarian. Russian and English summaries)

Hungarian version of a paper in Acta Sci. Math. Szeged 15 (1954), 211-222; MR 16, 327.

Billimovitch, Anton. Divector and its algebra. Glas Srpske Akad. Nauka 206. Od. Prirod.-Mat. Nauka (N.S.) 5 (1953), 57-70. (Serbo-Croatian. English summary)

The complex of two vectors, one free and one sliding, has been the object of the investigations of many scientists (J. Plücker, R. Ball, F. Klein, A. B. Kotelnikov and F. Study), under different names and with more or less different interpretations. But all these interpretations bore an intuitive character closely connected to concrete problems taken from geometry and mechanics. Only R. von Mises [Z. Angew. Math. Mech. 4 (1924), 155-181, 193-213] has built up his "motor" calculus (Motorrechnung) on a more abstract ground. But his conception too was not completely free of geometrical elements and above all his "motor" unity was rather unsatisfying. The author of the present paper considers under the name "divector" an abstract entity consisting of two vectors of different nature but without any connection with sliding vectors or any concrete notions. He gives the abstract algebraic operations for this entity, using in the development his own definition of the "divector" unity [ibid. 14 (1934), 189] which is different from that given by von Mises.

T. P. Andelić (Belgrade).

Watson, G. N. Two more tripos questions. *Math. Gaz.* 39 (1955), 280-286.

Two problems involving complicated but ingenious cubic identities were included in the Tripos papers for the 5th and 19th January, 1881. The author gives his own solutions of these problems, pointing out that the first is a fairly simple consequence of the second. He also reviews previous solutions and makes some illuminating comments and conjectures about their history and about the examiners who were responsible for setting, inadvertently or otherwise, two cognate questions neither of which was likely to receive many satisfactory answers, if any.

W. Ledermann (Manchester).

Gonçalves, J. Vicente. Sur l'élimination. Univ. Lisboa. *Revista Fac. Ci. A.* (2) 3 (1954-1955), 311-316.

A new version is given of a necessary and sufficient condition for two polynomials $f(x)$ and $g(x)$ to have a highest common factor D of degree β . The method is based on the construction of certain polynomials $\Phi = A/+Bg$, where A and B are suitable polynomials. Determinants formed from the coefficients of the Φ lead to an explicit expression for D . The author's result contains as special cases theorems on elimination by Sylvester, Osorio, Bézout and Cauchy.

W. Ledermann.

Sokolov, N. P. Affine projective classification of cubic ternary forms in the real domain. *Dopovidi Akad. Nauk Ukrain. RSR* 1955, 315-317. (Ukrainian. Russian summary)

This paper quotes the results obtained by the author using the three-dimensional matrices for the construction of affine-projective absolute invariants of the ternary cubic forms. A more detailed discussion is reserved for other papers. The author gives nine absolute affine invariants without discussion of the question whether this system is complete, reduced or a smallest system of invariants.

E. M. Bruins (Amsterdam).

See also: Mordell, p. 715; Schell, p. 760; Ghika, p. 767; Tucker, p. 778.

Lattices

Richardson, Moses. Solutions of irreflexive relations. *Ann. of Math.* (2) 58 (1953), 573-590; errata 60 (1954), 595.

Soit \mathcal{D} un ensemble, $G_0 \subset \mathcal{D} \times \mathcal{D}$ une relation binaire irreflexive (c'est à dire telle que $G_0 \cap \Delta = \emptyset$) où Δ est la diagonale de \mathcal{D}). Une solution de G_0 est, par définition, un sous ensemble V de \mathcal{D} tel que $x \in V$ et $y \in V$ implique $(x, y) \notin G_0$, et $y \notin V$ implique l'existence de $x \in V$ tel que $(y, x) \in G_0$. Ce concept de solution a été introduit par von Neumann et Morgenstern [Theory of games and economic behavior, Princeton, 1944; MR 6, 235]. L'auteur a montré [Bull. Amer. Math. Soc. 52 (1946), 113-116; MR 7, 235] que, lorsque $G_0 \cup G_0^{-1}$ ne contient pas de cycle impair, alors G_0 admet au moins une solution. Mais ce résultat est de peu d'utilité en théorie des jeux, car l'hypothèse que $G_0 \cup G_0^{-1}$ ne contient pas de cycle impair implique que G_0 est intransitive. On établit ici l'existence de solutions dans certains cas où l'intransitivité n'est pas demandée. Ainsi, le théorème 1 s'énonce: Si G_0 est fini et ne contient pas de cycle impair, alors G_0 admet une solution. Les théorèmes 2 et 3 apportent quelques modifications à ce théorème.

La considération du cas où G_0 est infini amène l'auteur à démontrer une série de théorèmes pour la validité desquels on doit supposer que G_0 est borné de diverses manières. C'est ainsi que le théorème 5 s'énonce: Si G_0 est régressivement bornée (c'est à dire telle qu'en chacun de ses points x il existe une borne pour la longueur des chaînes descendantes partant de x) et si G_0 ne contient pas de cycle impair, alors G_0 admet au moins une solution. Ce théorème est généralisé (th. 10) sous la forme suivante: Si G_0 satisfait à la condition des chaînes descendantes (ce qui, dans le langage de l'auteur s'énonce: Si G_0 est régressivement fini) et si G_0 ne contient pas de cycle impair, alors G admet au moins une solution.

J. Rignot (Paris).

Richardson, Moses. Extension theorems for solutions of irreflexive relations. *Proc. Nat. Acad. Sci. U.S.A.* 39 (1953), 649-655.

Etude de conditions suffisantes pour garantir l'existence d'une solution de $G \subset \mathcal{D} \times \mathcal{D}$ lorsqu'on a établi l'existence d'une solution de $G_0 = G \cap (\mathcal{D}_0 \times \mathcal{D}_0)$ où $\mathcal{D}_0 \subset \mathcal{D}$ (voir analyse précédente). Les théorèmes 2 à 11 sont des compléments ou des variantes du théorème 1 dont l'énoncé est trop long pour pouvoir figurer ici. Le rapporteur pense que l'emploi du calcul des relations binaires permettrait de simplifier énoncés et démonstrations.

J. Rignot (Paris).

Kamel, Hyman. Relational algebra and uniform spaces. *J. London Math. Soc.* 29 (1954), 342-344.

The author calls "relational algebra" a boolean algebra with greatest element V and with least element Λ , on which is defined one binary operation and one unary operation satisfying the following axioms: $(R \cdot S) \cdot T = R \cdot (S \cdot T)$; there exists I such that $R \cdot I = R$; $(R^-)^- = R$; $(RS)^- = S^- R^-$; $R \cdot \Lambda = \Lambda R = \Lambda$; $(R \cdot S) \cap TC(R \cap (T \cdot S^-)) \cdot (S \cap (R^- \cdot T))$. He shows that this system of axioms is independent and gives rise to the calculus of binary relations. He calls "uniform relational algebra" a relational algebra which contains a subset \mathfrak{A} such that \mathfrak{A} is a "filtre d'entourages" in Bourbaki's sense, and shows that when \mathfrak{A} is a complete boolean algebra the closure operation $\bar{R} = \bigcap_{A \in \mathfrak{A}} A \cdot R \cdot A$ is such that $\bar{R} = \bar{R}$, $\overline{R \cup S} = \bar{R} \cup \bar{S}$. Some standard results of uniform-space theory may be deduced algebraically.

J. Rignot (Paris).

Schützenberger, Marcel Paul. Théorie combinatoire des relations bilinéaires classiques. II. *Bull. Sci. Math.* (2) 79 (1955), 111-128.

This is a continuation of the author's previous study [same Bull. 79 (1955), 12-32; MR 16, 990] of abstract "classic bilinear relations". Sufficient conditions are given for the lattice of closed subsets to be modular or semi-modular. "Acentral", "orthogonal", and "symplectic" elements are defined, and some of their properties deduced. The distinctions made correspond to differences between the formal properties of different kinds of classic bilinear relations.

G. Birkhoff (Cambridge, Mass.).

Gelfand, M. S. Segments in a Dedekind lattice. *Moskov. Gos. Ped. Inst. Uč. Zap.* 71 (1953), 199-204. (Russian)
In a lattice, say that c lies between a and b (written acb) if and only if

$$ac + bc = c = (a + c)(b + c);$$

[cf. Pitcher and Smiley, *Trans. Amer. Math. Soc.* 52 (1942), 95-114; MR 4, 87; Sholander, *Proc. Amer. Math.*

Soc. 3 (1952), 369-381; MR 14, 9]. Various equivalent conditions are obtained in the case of a modular or distributive lattice; some of these are in terms of union and intersection, some in terms of distributive elements, and some are in terms of segments, where the segment $[a, b]$ is defined as the set of all c with acb . Principal theorems: In a modular lattice, a distributive element u which has a complement satisfies the condition, which Glivenko [Amer. J. Math. 59 (1937), 941-956] found for distributive lattices, that uxa , uxb and acb together imply uxc , while uay , uby , and acb together imply ucy . Any segment $[a, b]$ in a modular lattice is a sublattice with 0 and 1. P. M. Whitman (Silver Spring, Md.).

Dwinger, Ph. Errata: On the group of automorphisms of the lattice of closure operators of a complete lattice. Nederl. Akad. Wetensch. Proc. Ser. A. 59=Indag. Math. 18 (1956), 128. See same Proc. 58 (1955), 507-511; MR 17, 450.

See also: Wang, p. 710; Nikodým, p. 720; Thoma, p. 767.

Rings, Fields, Algebras

Auslander, Maurice; and Buchsbaum, David A. Homological dimension in Noetherian rings. Proc. Nat. Acad. Sci. U.S.A. 42 (1956) 36-38.

Let M be a module over a ring R ; if, for some n , there exists a projective resolution (X_k) for M such that $X_{n+1}=0$, then the smallest n for which this happens is called the homological dimension of M ($h \dim M$); if no such n exists, then $h \dim M$ is defined to be ∞ . The global dimension of R ($gl \dim R$) is defined to be the supremum of the homological dimensions of all R -modules. The authors announce the following result: a local ring R (i.e. a commutative Noetherian ring R with a unique maximal prime ideal) is regular if and only if it is of finite global dimension; and its global dimension is then equal to its dimension. It follows from this that, if R is a regular local ring, then so is R_P for every prime ideal P in R . The authors also establish that every regular local ring of dimension 2 is a unique factorization domain.

C. Chevalley (Paris).

Demaria, Davide Carlo. Sulla definizione di corpo. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 18 (1955), 268-274.

The paper contains another proof of the theorem that the usual postulates defining an associative division ring are not independent. The author's choice of postulates is as follows. Let F be a set of elements closed under two distinct binary operations, denoted, as usual, by addition and multiplication. Assume that F contains an element 0 such that $0+0=0$, $0 \cdot 0=0$, and let F^* be $F-0$. If F^* is a group under multiplication, with unit e , if there exists $-e$ in F such that $-e+e=0$, if $0+e=e$, if $y(x+e)=yx+y$ and $(e+x)y=y+xy$ for all x in F and y in F^* , and if $e+(x+y)=(e+x)+y$ for all x and y in F^* , then F is an associative division ring (and conversely, of course).

R. L. San Soucie (Eugene, Ore.).

Behrens, Ernst-August. Zur topologischen Darstellung nichtassoziativer Ringe. Arch. Math. 7 (1956), 41-48.

The author obtains a non-associative generalization of some results of Arens and Kaplansky [Trans. Amer. Math.

Soc. 63 (1948), 457-481; MR 10, 7]. Specially noteworthy is the final theorem, which gives a representation of suitable alternative algebras by functions in the Cayley numbers, successively restricted on closed subsets to quaternions, complex numbers, and real numbers.

I. Kaplansky (Chicago, Ill.).

Beyer, Gudrun. Über die Einbettung zyklischer Körper in metazyklische. Abh. Math. Sem. Univ. Hamburg 19 (1955), 127-133.

Let Ω be a normal extension field of the ground field Ω_0 . Hasse [Math. Nachr. 1 (1948), 40-61, 213-217, 277-283; MR 10, 426, 503] studied the problem of imbedding Ω into a Galois algebra K over Ω_0 with a given Galois group. His investigations led to a conjecture which however was shown to be false by Faddeev [Dokl. Akad. Nauk SSSR (N.S.) 94 (1954), 1013-1016; MR 15, 938] and Šafarevič [ibid. 95 (1954), 459-461; MR 16, 572]. The author shows that Hasse's conjecture is correct in the special case where Ω/Ω_0 is cyclic and where the condition on the Galois group of K implies that K/Ω is cyclic. As in Hasse's paper, it is assumed that Ω contains the n th roots of unity and that the characteristic of Ω is prime to n where n is the exponent of the group which is to be the Galois group of K/Ω . In the case treated here, this means that n is the order of this group.

R. Brauer (Cambridge, Mass.).

Deprit, André. A. S. Eddington's E -numbers. Ann. Soc. Sci. Bruxelles. Sér. I. 69 (1955), 50-78.

The E -numbers of Eddington are the elements of a 16-dimensional algebra over the field of complex numbers, isomorphic to the algebra of matrices of degree 4. The tensor product of two simple algebras over the complex numbers being simple, the algebra of E -numbers may be considered to be the tensor product of two quaternion algebras. The author calls generalized sedenion algebra an algebra (over any commutative ring) which is the tensor product of two generalized quaternion algebras. Since the generalized quaternion algebras over a field K of characteristic $\neq 2$ are the Clifford algebras of quadratic forms of dimension 2, the sedenion algebras over K may be interpreted as Clifford algebras of quadratic forms of dimension 4; for the same reason, a sedenion algebra may be regarded as the algebra of even elements of the Clifford algebra of a quadratic form of dimension 5. The author shows that sedenion algebras may also be related to the Clifford algebras of quadratic forms of dimension 6; in this connexion, the reader should not be misguided by the false notation $C_2(E) \cap CC_1(E)$ used by the author on p. 76: what is meant is the subspace of $C(E)$ spanned by e_0 and by the elements $e_i e_j$ ($0 < i < j$); it is also erroneous to say that this space, with the multiplication on it defined by the author, is a subalgebra of $C(E)$.

Let Q be the ordinary quaternion algebra, with its base (e_0, e_1, e_2, e_3) , e_0 being the neutral element and $e_i^2 = -e_0$ ($i=1, 2, 3$), $e_1 e_2 = e_3$. The elements $\pm e_i \otimes e_j$ of $Q \otimes Q$ form a multiplicative group S , which the author calls the Dirac group. He classifies all subgroups of S , and he determines the structure of the group algebra of S .

C. Chevalley (Paris).

Kawahara, Yūsaku. On the derivations in maximal orders of simple algebras. Nagoya Math. J. 9 (1955), 147-160.

The author extends the results of Kawada [Sci. Papers Coll. Gen. Ed. Univ. Tokyo 2 (1952), 1-8; MR 14, 348]

to the case of derivations of maximal orders of simple algebras over fields which are quotient fields of arbitrary Dedekind rings; the new feature in that case is that there may exist nontrivial unramified division algebras over the completions of the field; nevertheless, the results for derivations remain the same as in the case of algebras over fields of algebraic numbers.

C. Chevalley (Paris).

See also: Wagner, p. 777.

Groups, Generalized Groups

Novikov, P. S. Unsolvability of the conjugacy problem in the theory of groups. *Izv. Akad. Nauk SSSR. Ser. Mat.* 18 (1954), 485-524. (Russian)

In a previous note [Dokl. Akad. Nauk SSSR (N.S.) 85 (1952), 709-712; MR 14, 618] the author has sketched a proof of the algorithmic unsolvability of the word problem in groups. A full proof appears in the monograph reviewed below. Now it follows trivially that in the group constructed there the conjugacy problem is also unsolvable. Nevertheless the present paper contains a separate proof of this result, because it is considerably simpler and because the number of generators and defining relations for the group with an unsolvable conjugacy problem is much smaller than for the group with an unsolvable word problem. The present construction is again based on Post's systems of "productions" [Amer. J. Math. 65 (1943), 197-215; MR 4, 209] which are transformations of the schemes: $A_i X \rightarrow X B_i$ and $X B_i \rightarrow A_i X$, where X is an arbitrary word in the alphabet under consideration, and where (A_i, B_i) are a finite number of preassigned pairs of fixed words in the alphabet. These transformations introduce an equivalence relation into the set of words which will be denoted as an equality. Now Post [loc. cit; see also Bull. Amer. Math. Soc. 52 (1946), 264-268; MR 7, 405; and Markov, Dokl. Akad. Nauk SSSR (N.S.) 58 (1947), 353-356; MR 9, 321] has constructed a system of productions for which there exists no algorithm to decide whether two given words are equal in the system or not. The link between the Post systems and groups is established by the following theorem: Whatever system of productions \mathfrak{P} is given, there exists a group \mathfrak{A} with a finite number of generators and of defining relations such that with every word X in \mathfrak{P} there is associated a word $\Phi(X)$ in \mathfrak{A} , where the mapping Φ has the following properties: 1. The construction of $\Phi(X)$ from X is algorithmic. 2. Two words $\Phi(X)$ and $\Phi(Y)$ are conjugate in \mathfrak{A} if and only if X and Y are equal in \mathfrak{P} . The construction of \mathfrak{A} on the basis of \mathfrak{P} is algorithmically effective, but the intricate combinatorial details of the process which make up the bulk of the paper are, unfortunately, too numerous to be even sketched here. Finally, for the systems \mathfrak{P} with a known unsolvable equality problem the associated groups \mathfrak{A} have an unsolvable conjugacy problem. The author mentions a consequence of his result: Since every finitely presented group is the fundamental group of a two-dimensional polyhedron, and since the homotopy problem for the polyhedron is equivalent to the conjugacy problem in the group, it is possible to construct a two-dimensional polyhedron for which the homotopy problem is unsolvable.

K. A. Hirsch (London).

★ Novikov, P. S. Ob algoritmičeskoj nerazrešimosti problemy toždestva slov v teorii grupp. [On the algorithmic unsolvability of the word problem in group theory.] *Trudy Mat. Inst. im. Steklov.* no. 44. Izdat. Akad. Nauk SSSR, Moscow, 1955, 143 pp. 6.30 rubles.

The following theorem is proved: There exists a group defined by a finite system of relations between the generators (the number of which is also finite) such that it is impossible to find an algorithm for deciding the identity of group elements given by products of powers of the generators.

An essential rôle in the proof is played by calculi of various types. Each of them deals with words in some alphabet and allows one to derive according to definite rules of procedure the equality of such words. (The equality of words does not always turn out to be symmetric, but is always reflexive and transitive.) In particular the following calculi are used.

Semigroups (with cancellation). In an arbitrary alphabet A there is given a system of pairs of words A_i, B_i ($i = 1, \dots, n$). The equations

$$(1) \quad A_i = B_i \quad (i = 1, \dots, n)$$

are taken as the initial ones. Other equations are subsequently derived from them according to the rules

$$\frac{u=v}{XuY=XvY}, \quad \frac{Xu=Xv}{u=v}, \quad \frac{uX=vX}{u=v}, \quad \frac{u=v, v=w}{u=w}.$$

Here u, v, X, Y are words in the alphabet A . Each rule allows one to derive the equation under the horizontal bar from the equation (or equations) above this bar. We shall call the resulting calculus a semigroup on the alphabet A , defined by the equations (1).

Groups. Two alphabets with the same number of letters are given: a "positive" alphabet A and a "negative" alphabet A^- . A one-to-one correspondence is set up between them. The letter of the alphabet A^- corresponding to a letter ξ of the alphabet A is called the inverse element to ξ and is designated by ξ^{-1} . A system of word pairs A_i, B_i ($i = 1, \dots, n$) is given in the alphabet $A \cup A^-$. One says that the word v is obtained from the word u by an elementary step if there are words X and Y in $A \cup A^-$ such that at least one of the following six conditions is satisfied:

- 1) there is an i such that $u = X A_i Y, v = X B_i Y$;
- 2) there is an i such that $u = X B_i Y, v = X A_i Y$;
- 3) there is a letter ξ in A such that $u = X \xi \xi^{-1} Y, v = XY$;
- 4) there is a letter ξ in A such that $u = X \xi^{-1} \xi Y, v = XY$;
- 5) there is a letter ξ in A such that $u = XY, v = X \xi \xi^{-1} Y$;
- 6) there is a letter ξ in A such that $u = XY, v = X \xi^{-1} \xi Y$.

Here the symbol "=" indicates identity of the words. The equality $u=v$ is considered as derivable if there exists a series of words u_0, \dots, u_m ($m \geq 0$) such that $u = u_0, v = u_m$ and such that for $0 < i \leq m$ the word u_i is obtained from the word u_{i-1} by an elementary step. We shall call the calculus obtained in this way a group on the alphabet $A \cup A^-$, defined by the relations (1).

Systems of type B. Their definition differs from that of groups only in that condition 6) is discarded in the definition of an elementary step.

Establishment of a series of connections between the calculi of various types plays an essential rôle. This makes it possible to reduce problems of deducibility of equations

in calculi of one type to problems of deducibility of equations in appropriate calculi of another type. As a result one succeeds in constructing for an arbitrary semigroup \mathfrak{R} a group \mathfrak{R}_1 such that the problem of deducibility of equations in \mathfrak{R} is reduced to the problem of deducibility of equations in \mathfrak{R}_1 . If one now takes for \mathfrak{R} the semigroup constructed by Turing [Ann. of Math. (2) 52 (1950), 491-505; MR 12, 239] for which the problem of deducibility of equations is unsolvable, then \mathfrak{R}_1 is a group with an unsolvable "word problem". These constructions run as follows.

Let \mathfrak{R} be a semigroup on the alphabet A defined by the relations (1). Let us construct a calculus \mathfrak{R}^0 of type B in the following manner. Each letter ξ of A is made to correspond to a letter ξ which we shall call the image of ξ . The images must be different from the letters of A and from each other. They form an alphabet of images \bar{A} containing the same number of letters as A . We also introduce a letter τ not belonging to the alphabet $A \cup \bar{A}$. As a positive alphabet in the calculus \mathfrak{R}^0 we take an alphabet B equal to $A \cup \bar{A} \cup \{\tau\}$. A negative alphabet B^- of the calculus \mathfrak{R}^0 is introduced by setting up for each letter η of B a new letter η^{-1} . As a system of defining relations we take the system consisting of all equations (1) and all equations

$$\xi\tau = \xi\tau, \quad \xi\eta = \eta\xi,$$

where ξ and η are letters of the alphabet A . There is the following connection between the problems of deducibility of equations in the calculi \mathfrak{R} and \mathfrak{R}^0 (Theorem 4 of chap. VI).

I. If u and v are words in the alphabet A , then the equation $u=v$ is deducible in \mathfrak{R} if and only if $u\tau=v\tau$ is deducible in \mathfrak{R}^0 .

Let \mathfrak{B} be a calculus of type B with a pair of alphabets Γ, Γ^- defined by the equations

$$(2) \quad C_i = D_i \quad (i=1, \dots, r),$$

where C_i and D_i are words in the alphabet Γ . Let us construct a second calculus \mathfrak{B}' of type B in the following manner. We introduce the letters

$$(3) \quad \lambda_i, \nu_i, d_i, \mu_{1i}, \mu_{2i}, \mu_{3i}, \mu_{4i}, \varrho_i, \tilde{\varrho}_i, l_{\xi}, p \quad (i=1, \dots, r; \xi \in \Gamma),$$

different from each other and not belonging to the alphabet $\Gamma \cup \Gamma^-$. The letters l_{ξ} are introduced for each letter ξ of the alphabet Γ and for each i from 1 to r . Let us now introduce for each of the new letters η an inverse letter η^{-1} such that these letters are different from each other, from the letters of the alphabet $\Gamma \cup \Gamma^-$ and from all letters in (3). The letters of the alphabet $\Gamma \cup \Gamma^-$ and all the introduced letters form the alphabet of the calculus \mathfrak{B}' and split into positive and negative alphabets $\bar{\Delta}$ and $\bar{\Delta}^-$. This calculus is defined by the equations.

$$(4) \quad d\lambda_i C_i = \nu\mu_{1i}\varrho_i d\mu_{2i}, \quad d\lambda_i D_i = \nu\mu_{1i}\tilde{\varrho}_i d\mu_{3i},$$

$$(5) \quad d_i \xi = \xi d_i,$$

$$(6) \quad \mu_{1i} \xi = \xi \mu_{1i} l_{\xi}, \quad \tilde{\mu}_{1i} \xi = \xi \tilde{\mu}_{1i} l_{\xi}$$

$$(7) \quad l_{\xi} \eta = \eta l_{\xi}$$

$$(8) \quad \mu_{2i} \xi = \xi l_{\xi} \mu_{2i}, \quad \mu_{3i} \xi = \xi l_{\xi} \tilde{\mu}_{3i},$$

$$(9) \quad \varrho_i \xi = \xi \varrho_i, \quad \tilde{\varrho}_i \xi = \xi \tilde{\varrho}_i,$$

$$(10) \quad l_{\xi} \mu_{2i} p = \mu_{2i} p, \quad l_{\xi} \mu_{3i} p = \tilde{\mu}_{3i} p,$$

$$(11) \quad l_{\xi} d\mu_{2i} p = d\mu_{2i} p, \quad l_{\xi} d\tilde{\mu}_{3i} p = d\tilde{\mu}_{3i} p,$$

$$(12) \quad \varrho_i d\mu_{2i} p = d\mu_{2i} p, \quad \tilde{\varrho}_i d\tilde{\mu}_{3i} p = d\tilde{\mu}_{3i} p,$$

$$(13) \quad \mu_{1i} d\mu_{2i} p = \tilde{\mu}_{1i} d\tilde{\mu}_{3i} p,$$

where i takes the values $1, \dots, r$ and ξ and η are arbitrary letters of Γ . The following connection holds between the problems of deducibility of equations in the calculi \mathfrak{B} and \mathfrak{B}' (Theorem 1 of chap. VI).

II. If u and v are words in the alphabet Γ , then the equation $u=v$ is deducible in the calculus \mathfrak{B} if and only if $u p = v p$ is deducible in \mathfrak{B}' . Let us further construct a group \mathfrak{B}^+ in the following manner. For each letter ξ of $\bar{\Delta}$, different from p , we introduce a new letter ξ^+ which we call the dual of ξ . Moreover, different letters are to have different duals. By the dual of a word X in the alphabet $\bar{\Delta} \setminus \{p\}$ we shall mean the word obtained from X by inverting the order of the letters and replacing all letters by their duals. By adjoining to $\bar{\Delta}$ the duals of all its letters other than p , we obtain the positive alphabet E of the group \mathfrak{B}^+ . We also introduce the letters ξ^{+-1} inverse to the duals. Adjoining them to the alphabet $\bar{\Delta}^-$, we obtain the negative alphabet E^- of the group \mathfrak{B}^+ . This group is defined by a system of equations obtained in the following way: to equations (4)-(9) adjoin the equations obtained from them by replacing both sides of each equation by their duals; then (in place of equations (10)-(13)) adjoin the equations

$$l_{\xi} \mu_{2i} p \mu_{2i}^+ l_{\xi}^+ = \mu_{2i} p \mu_{2i}^+, \quad l_{\xi} \tilde{\mu}_{3i} p \tilde{\mu}_{3i}^+ l_{\xi}^+ = \tilde{\mu}_{3i} p \tilde{\mu}_{3i}^+,$$

$$l_{\xi} d\mu_{2i} p \mu_{2i}^+ d_i^+ l_{\xi}^+ = d\mu_{2i} p \mu_{2i}^+ d_i^+,$$

$$l_{\xi} d\tilde{\mu}_{3i} p \tilde{\mu}_{3i}^+ d_i^+ l_{\xi}^+ = d\tilde{\mu}_{3i} p \tilde{\mu}_{3i}^+ d_i^+,$$

$$\varrho_i d\mu_{2i} p \mu_{2i}^+ \varrho_i^+ = d\mu_{2i} p \mu_{2i}^+,$$

$$\tilde{\varrho}_i d\tilde{\mu}_{3i} p \tilde{\mu}_{3i}^+ \tilde{\varrho}_i^+ = d\tilde{\mu}_{3i} p \tilde{\mu}_{3i}^+,$$

$$\mu_{1i} d\mu_{2i}^+ d_i^+ \mu_{1i}^+ = \tilde{\mu}_{1i} d\tilde{\mu}_{3i}^+ \tilde{\mu}_{1i}^+ d_i^+.$$

We now construct the group \mathfrak{B}^{++} from the calculus \mathfrak{B}' in the same way that the group \mathfrak{B}^+ was obtained from the calculus \mathfrak{B} . This means that the role of the positive alphabet of the calculus \mathfrak{B}' will be played by the positive alphabet of the calculus \mathfrak{B} ; the role of equations (2) by equations (4)-(13), the right and left sides of which are words in the alphabet $\bar{\Delta}$; the role of r by the number of the equations (4)-(13), which we designate by r^* . The role of the letters (3) must now be played by the new letters

$$\lambda_i^*, \nu_i^*, d_i^*, \mu_{1i}^*, \mu_{2i}^*, \mu_{3i}^*, \mu_{4i}^*, \varrho_i^*, \tilde{\varrho}_i^*, l_{\xi}^*, p^* \quad (i=1, \dots, r^*; \xi \in \bar{\Delta}),$$

not belonging to the alphabet $\bar{\Delta}$.

The following connection holds between the problems of deducibility of equations in the calculi \mathfrak{B}' and \mathfrak{B}^{++} (Theorem 2 of chap. VI).

III. If u and v are words in $\bar{\Delta}$, then the equation $u=v$ is deducible in the system \mathfrak{B}' if and only if $u p^* u^+ = v p^* v^+$ is deducible in the group \mathfrak{B}^{++} .

Theorems I, II, and III make possible the following method of proving the main result of the work.

Starting from Turing's semigroup \mathfrak{R} for which the problem of deducibility of equations is not solvable, we construct a system \mathfrak{R}^0 of type B. Taking for Γ the positive alphabet of the system \mathfrak{R}^0 and remarking that \mathfrak{R}^0 is defined by a system of equations whose right and left sides are words in Γ , we write these equations in the form (2) and construct on \mathfrak{R}^0 the system $\mathfrak{R}^{0'}$ and the group \mathfrak{R}^{0++} . From Theorems I, II, and III there follows the truth of the following assertion.

If u and v are words in the alphabet of the semigroup \mathfrak{R} , then $u=v$ is deducible in \mathfrak{R} if and only if the equation

$$u \tau p p^* p^+ \tau^+ u^+ = v \tau p p^* p^+ \tau^+ v^+$$

is deducible in the group \mathfrak{R}^{0++} .

With this the problem of deducibility of equations in

the semigroup \mathfrak{R} is reduced to the problem of deducibility of equations in the group \mathfrak{R}^{0+} . And since the first problem is unsolvable, the second is also unsolvable, as we wished to prove.

The exposition is rather inadequate, which makes for very difficult reading. There are numerous inaccuracies and some untrue assertions. In addition there are many misprints, of which only a part appear on the list of errata appended to the monograph. All these defects, however, do not affect the essence of the matter. After a careful study of the monograph the reviewer has come to the conclusion that the small errors in it can all be easily corrected, and hence that the result of this remarkable work is valid.

A. A. Markov (Moscow).

Tutte, W. T. A class of Abelian groups. *Canad. J. Math.* 8 (1956), 13-28.

If M is a finite set, a mapping f of M into the set of ordinary integers is called a chain on M . If $f(a)=0$ for all $a \in M$, then f is the zero chain on M . The sum $f+g$ of two chains f and g on M is a chain on M defined by $(f+g)(a)=f(a)+g(a)$ for each $a \in M$. With this definition of addition the chains on M are the elements of an additive Abelian group. A subgroup of this is called a chain-group on M .

The set of all $a \in M$ such that $f(a) \neq 0$ is called the domain $|f|$ of f . If N is a chain-group on M a chain f of N is elementary if it is non-zero and there is no nonzero $g \in N$ such that $|g| \subset |f|$. If in addition f takes only the values 0, 1, -1, or a subset of these, then f is called a primitive chain of N . N is called regular if for each elementary chain f of N there exists a primitive chain g of N such that $|g|=|f|$.

The properties of chain-groups, especially the regular ones, are investigated, chiefly from the point of view of the structure of finite oriented graphs. For example, it is shown that the set of all cycles of the graph is a regular chain-group on the set of edges, and so is the set of coboundaries of the 0-chains on the graph. A number of operations and definitions concerning chains are introduced, which are of interest in themselves and are also related to the theory of graphs. In the last section two graph-theoretical results are deduced: If G is an oriented graph, any edge of G is an edge of a directed bond or of a directed circuit of G . The second is a known theorem concerning the 1-factors of even graphs: If G is even and balanced and V_1 and V_2 are the two vertex-colour-classes, then G has a 1-factor if and only if there is no subset U of V_1 such that the set of all vertices of V_2 joined by edges of G to vertices of U has fewer members than U . [See, e.g., P. Hall, *J. London Math. Soc.* 10 (1935), 26-30.]

G. A. Dirac (Vienna).

Fuchs, László; and Szele, Tibor. Abelian groups with a single maximal subgroup. *Magyar Tud. Akad. Mat. Fiz. Oszt. Közl.* 5 (1955), 387-389. (Hungarian)

The authors prove that an abelian group G has a single maximal subgroup if and only if it admits a representation as a direct sum $A+B$, where A is an arbitrary complete abelian group, and B is either a group of prime-power order or a group isomorphic to a pure subgroup ("Servanzuntergruppe") of the additive group of the p -adic integers. This theorem answers an apparently difficult group-theoretical question in the commutative case.

A. Kertész (Debrecen).

Wielandt, Helmut. Primitive Permutationsgruppen vom Grad $2p$. *Math. Z.* 63 (1956), 478-485.

Suppose that p is a prime and that G is a primitive group of permutations of degree $2p$. Denote by G_0 the subgroup of all those permutations in G which have a certain cypher as fixed element; and denote by t the number of domains of transitivity of G_0 . Then $0 < t \leq 3$; and $t=3$ implies; $2p=m^2+1$ and the three domains of transitivity of G_0 have degrees 1, $\frac{1}{2}m(m-1)$ and $\frac{1}{2}m(m+1)$. This implies in particular that primitive groups of permutations of degree $2p \neq m^2+1$ are always doubly transitive. The proofs of these theorems make essential use of the theory of representations and group characters.

R. Baer (Urbana, Ill.).

Piccard, Sophie. Structure des groupes d'ordre fini jouissant de la propriété $P \pmod{p}$. *Bull. Sci. Math.* (2) 78 (1954), 240-262.

A group G of finite order N with a system of $m \geq 1$ independent generators is said to enjoy the property $P \pmod{p}$ if the degree in each S_i of each characteristic relation $F(S_1, S_2, \dots, S_m)=1$ is a multiple of p . This property is seen to be independent of the chosen basis of generators. In particular, the order of each generator is a multiple of p . Those elements of G having a representation as products of powers of the generators S_1 to S_m in which the exponent sum for S_i is a_i are said to form the class M_{a_1, a_2, \dots, a_m} . These classes are cosets of the characteristic subgroup $M_{0,0,\dots,0}$ which contains the commutator subgroup of G and also contains all elements of G whose orders are relatively prime to p . The factor group Γ is abelian of order p^m and type $(1, 1, \dots, 1)$. The image in Γ of any basis for G is one of the $B_m = (p^m-1)(p^m-p) \dots (p^m-p^{m-1})/m!$ bases of Γ . Hence G has at most $(n/p^m)m B_m$ bases, no one of whose basis elements is in $M_{0,0,\dots,0}$. To each subgroup of Γ corresponds a characteristic subgroup of G which is the inverse image of Γ . The author also discusses certain other collections C of cosets of $M_{0,0,\dots,0}$ corresponding to subgroups of order p^{m-1} of Γ and their cosets. It is shown that every abelian group ($N > 1$) has the property P for some prime p , and that there exist for every m and p non-abelian groups having this property.

J. S. Frame (East Lansing, Mich.).

Taussky, Olga. A note on group matrices. *Proc. Amer. Math. Soc.* 6 (1955), 984-986.

Let G be a finite group with n elements P, Q, \dots , and M the group matrix corresponding to G , i.e. the $n \times n$ matrix $(x_P x_Q^{-1})$, $P, Q \in G$, where x_P, x_Q, \dots are n indeterminates in 1-1 correspondence with the elements of G . The matrix M is called normal if $MM^* = M^*M$, where M^* is the transposed complex conjugate of M , conjugates being formally introduced for the indeterminates. The indeterminate x is considered as real if it coincides with its conjugate. The following theorem is proved. Let the indeterminates x be real, then M is normal if and only if G is an abelian or a hamiltonian 2-group. For complex x 's the matrix M is normal if and only if G is an abelian group.

A. Kertész (Debrecen).

Schiek, Helmut. Gruppen mit Relationen $X^3=1, (XY)^3=1$. *Arch. Math.* 6 (1955), 341-347.

Groups whose elements all satisfy the relation $X^3=1$ were studied in a paper by F. Levi and B. L. van der Waerden [*Abh. Math. Sem. Hamburg Univ.* 9 (1932), 154-158]. Here the author considers a group G whose generators in a given system S all satisfy relations of the form $X^3=1, (XY)^3=1$. Those products of generators in S for which the sum of the exponents is a multiple of 3

form a normal subgroup N . A system B of generators for N can be built from the products $X_1 = X^2 A$, $X_r = A X^2$, where A is a fixed generator in S and X runs through all the other generators in S . The relations among these generators of N may be written in the commutator form $(X_1^{-1}, X_r) = 1$, $(X_1^{-1}, Y_r)(Y_1^{-1}, X_r) = 1$. The author introduces an ordering among the elements X_r , and a corresponding induced ordering among the X_1 , and thus defines what he calls regular commutators and normal commutators. Each collection, regular or normal, is shown to form a system of generators for the commutator subgroup of N . G is obtained from N by extending N by an element A of order 3, and using the automorphisms $A X_1 A^{-1} = X_r$, $A X_r A^{-1} = X_1^{-1} = X_1^{-1} X_r^{-1}$, to define the extension.
J. S. Frame (East Lansing, Mich.).

Auslander, Maurice; and Lyndon, R. C. Commutator subgroups of free groups. Amer. J. Math. 77 (1955), 929-931.

Soient F un groupe libre, non-abélien; R un sous-groupe normal, $\neq F$ et $\neq (1)$; $G = F/R = F_0/R_0$, avec $F_0 = F/[R, R]$, $R_0 = R/[R, R]$. Les automorphismes intérieurs de F font opérer G dans R_0 . Théorème 1: G opère effectivement dans R_0 (la démonstration utilise le "cup product reduction theorem" d'Eilenberg-MacLane). Corollaire: le centre C_0 de F_0 est contenu dans R_0 et $\neq R_0$. Théorème 2: Pour que $C_0 \neq (1)$, il faut et il suffit que G soit fini. Théorème 3: Soient R et S des sous-groupes normaux d'un F libre non abélien; si $[S, S] \subset [R, R]$, alors SCR .

Note du rapporteur: la conclusion du th. 3 subsiste sous les hypothèses plus faibles: $[S, S] \subset CR$, $[S, S] \subset [R, R]$ et $[R, S] \subset [R, R]$. Dans le cas où $S = F$, on obtient le "lemme" utilisé par les auteurs pour prouver le théorème 3.
H. Cartan (Paris).

Scott, W. R. On infinite groups. Pacific J. Math. 5 (1955), 589-598.

The author proves several more or less disconnected theorems on infinite groups. In the first place he investigates the indices of subgroups and obtains among other results a generalization of Poincaré's theorem on the index of the intersection of a finite number of subgroups. A further theorem generalizes a previous result of the author [Lemma 1 in Amer. J. Math. 74 (1952), 187-197; MR 13, 721] on the layer of elements of infinite order in a group. It is shown that the subgroup K of an infinite group G (as defined in the paper cited above) is over-characteristic in the sense of B. H. Neumann and H. Neumann [Math. Nachr. 4 (1951), 106-125; MR 12, 671]. Finally, complete descriptions are given for classes of abelian groups with special properties, generalizing some results of Szele and Szélpál [Acta Univ. Sect. Szeged. Sci. Math. 13, 51-53, 54-56 (1949), 192-194 (1950); MR 11, 7; 12, 477].
A. Kertész (Debrecen).

Smirnov, D. M. On a class of infinite groups. Ivanov. Gos. Ped. Inst. Uč. Zap. Fiz.-Mat. Nauki 5 (1954), 57-60. (Russian)

In a group a finite normal series every infinite factor of which is commutative is called a B -series. A group having a B -series is called a B -group. The principal result states that every B -group is the extension of a solvable group by a semisimple finite group. Numerous properties are listed for B_i -groups, where, for $i = 1, 2, 3, 4, 5$, a group is called a B_i -group if it has a B -series, the abelian factors of which have type A_i in the sense of Mal'cev [Mat. Sb. N.S. 28(70) (1951), 567-588; MR 13, 203].
R. A. Good.

Szász, Ferenc. On groups of which all non-trivial powers are cyclic subgroups. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. 5 (1955), 491-492. (Hungarian)

Let G^k denote the subgroup of an arbitrary group G generated by all elements of the form $g^k \in G$, where k is a rational integer. If $k \neq 0, 1, -1$, G^k is called a non-trivial power of G . It is shown that a group is cyclic if and only if every non-trivial power of it is cyclic.
A. Kertész.

Plotkin, B. I. Radical groups. Mat. Sb. N.S. 37(79) (1955), 507-526. (Russian)

A group possessing a property θ is called a θ -group; θ is any local property pertaining to groups such that every subgroup and every homomorphic image of a θ -group are also θ -groups and such that every group has a θ -radical, that is, a characteristic θ -subgroup containing all the invariant θ -subgroups of the group. In a group G an ascending (possibly transfinite) series of characteristic subgroups $\theta_\alpha(G)$, with $\theta_0(G) = E$, is a θ -radical series when, in the case of the non-limiting ordinal $\alpha + 1$, $\theta_{\alpha+1}(G)/\theta_\alpha(G)$ is the θ -radical of $G/\theta_\alpha(G)$. The upper θ -radical of G is the first term of the θ -radical series which equals its successor; G is called a θ -radical group provided it equals its upper θ -radical. If a group has an ascending normal (or invariant) series whose first term is a θ -subgroup, then the latter is contained in the non-trivial θ -radical of the group. A group G is a θ -radical group if G has an ascending normal series from E to G , all factors of which are θ -groups. The upper θ -radical of a group G is the intersection of all normal subgroups H such that G/H is θ -semisimple. In the case of the most intensively investigated illustration of a property θ , namely local nilpotency, the prefix θ is dropped from θ -radical. In a radical group the radical contains its own centralizer. In a radical group the set of all nil-elements forms a subgroup which coincides with the radical of the group. A radical group of matrices is solvable. If the radical R of a radical group G is a nilpotent torsion-free group of finite rank, then G is solvable and has an invariant series $ECRCHCG$, where H/R is abelian and G/H is a finite radical group. If G is a radical group of finite special rank, then G has a series of normal subgroups $ECH_1CH_2CH_3CG$, where H_1 is the maximal periodic normal subgroup of G , H_2/H_1 is a nilpotent torsion-free group of finite rank, H_3/H_2 is abelian, and G/H_3 is finite. If every abelian subgroup of the radical of a radical group is finite, the group itself is finite. In a group of finite special rank the upper radical is an RJ^* -group. Several conditions are cited as sufficient that an R^* -group be a radical group and be the extension of its radical by a torsion-free abelian group.
R. A. Good.

Kochendörffer, Rudolf. Über den Multiplikator einer Gruppe. Math. Z. 63 (1956), 507-513.

The multiplier (Multiplikator) of a group \mathfrak{G} was first considered by Schur [J. Reine Angew. Math. 127 (1904), 20-50] and may be defined as the first cohomology group of \mathfrak{G} relative to the multiplicative group of a given field \mathfrak{F} on which \mathfrak{G} acts trivially. The author is concerned with the case in which \mathfrak{G} is finite and \mathfrak{F} is algebraically closed and of characteristic not dividing the order of \mathfrak{G} . Under these hypotheses, he proves the following two theorems, the first of which is due to Schur [loc. cit.]. (1) If p is a prime dividing the order of \mathfrak{G} , then a p -Sylow subgroup of the multiplier of \mathfrak{G} is isomorphic to a subgroup of the multiplier of a p -Sylow subgroup of \mathfrak{G} . (2) Let \mathfrak{E} be a p -Sylow subgroup of \mathfrak{G} and let \mathfrak{B} be a complete system of right coset representatives of \mathfrak{G} relative to \mathfrak{E} such that

$P^{-1}\mathfrak{S}P=\mathfrak{S}$ for each P in \mathfrak{S} . Then a p -Sylow subgroup of the multiplier of \mathfrak{S} is isomorphic to the multiplier of the p -Sylow subgroup \mathfrak{S} of \mathfrak{S} .

R. Steinberg.

Lazard, Michel. Sur la nilpotence de certains groupes algébriques. C. R. Acad. Sci. Paris 241 (1955), 1687-1689.

Let A be a commutative ring with unit element, and let f_1, \dots, f_n be n polynomials with coefficients in A in $2n$ undetermined. If B is any commutative unitary algebra over A , then f_1, \dots, f_n define in an obvious manner a polynomial law of composition in the set B^* . If, for any such B , B^* is a group with respect to this law of composition, then (f_1, \dots, f_n) is called a polynomial group law. This being the case, the author proves that, for any choice of B , B^* is necessarily a nilpotent group, which is furthermore of class $\leq n$ if A has no nilpotent element $\neq 0$ (this had been conjectured in a weaker form by P. Cartier). Considering the ring generated by the coefficients of the polynomials f_i , the proof is reduced to the case where A is a finitely generated ring; reducing modulo the maximal prime ideals of A , one sees that it is sufficient to prove the theorem in the case where A is a finite field. In that case, the result is deduced by a clever trick from the fact that, if B is a finite overfield of A , then the group B^* is nilpotent because it is a p -group (p being the characteristic of A). Formula " $W=C_2(X, W)$ " on the third line before the end of the proof of Lemma 1 should read " $W=C_2(X, V)$ ".

C. Chevalley (Paris).

★ **Boerner, Hermann.** Darstellungen von Gruppen mit Berücksichtigung der Bedürfnisse der modernen Physik. Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, Bd LXXIV. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1955. xi+287 pp. DM 36.60.

The contents of this book can be compared roughly to those of H. Weyl's, "Classical groups, their invariants and representations", [Princeton, 1939; MR 1, 42], F. D. Murnaghan's, "The theory of group representations" [Johns Hopkins Press, Baltimore, 1938], and D. E. Littlewood's "The theory of group characters and matrix representations of groups" [2nd ed., Oxford, 1950; for a review of the 1st ed. see MR 2, 3]. It is the first book in German along those lines. It was the wish of the author to be understandable to physicists. For this reason, he tried to make the book as self-contained as possible, and he did not attempt to treat ideas such as integration on groups or Lie algebras from a modern mathematical point of view. The main interest centers on special groups, the symmetric and the alternating groups, the full linear and the unimodular groups, the unitary groups, and the rotation groups. The treatment of the latter is of special interest; the method used is that of E. Stiefel [Comment. Math. Helv. 14 (1942), 350-380; 17 (1945), 165-200; MR 4, 134; 7, 115].

The following list of chapters will give an idea of the contents. I. Matrices. II. Groups. III. General theory of representation. IV. The representations of the symmetric groups. V. The representations of the full linear, unimodular, unitary groups. VI. Characters of the full linear, the symmetric and the alternating groups. VII. Characters and single-valued representations of the rotation groups. VIII. Spin representations and the Lie algebras of the rotation groups. IX. The Lorentz group.

It will be clear that, because of the self-imposed limita-

tions, the book is of greater interest to a reader studying the special groups than to one interested in the general aspects of the theory. A particular attraction of the treatment of these special groups lies in the fact that the reader is made familiar with entirely different methods by which these groups can be studied. A great number of interesting details are given. However, it is not possible to describe them here. R. Brauer (Cambridge, Mass.).

Reiner, Irving. Real linear characters of the symplectic modular group. Proc. Amer. Math. Soc. 6 (1955), 987-990.

Let Γ_{2n} denote the symplectic modular group of $2n \times 2n$ matrices with integer elements. Let K_{2n} denote the commutator subgroup of Γ_{2n} . It is proved by the author that $K_{2n}=\Gamma_{2n}$ for $n>2$; $\Gamma_4:K_4=2$ and $\Gamma_2:K_2=12$, and the factor groups $\Gamma_4:K_4$ and $\Gamma_2:K_2$ have been determined.

L. K. Hua (Peking).

Murnaghan, Francis D. On the Kronecker product of irreducible representations of the symmetric group. Proc. Nat. Acad. Sci. U.S.A. 42 (1956), 95-98.

The representation of the symmetric group on n symbols corresponding to the partition $[n-p, \mu_1, \mu_2, \dots, \mu_r]$ is denoted by $(\mu)=(\mu_1, \dots, \mu_r)$ and is associated with the S-function $\{\mu\}$.

It is known that

$$\{\mu\}S_r = \{\mu_1+r, \mu_2, \dots, \mu_r\} + \{\mu_1, \mu_2+r, \dots, \mu_r\} + \dots$$

There is a corresponding quotient

$$\{\mu/S_r\} = \{\mu_1-r, \mu_2, \dots, \mu_r\} + \{\mu_1, \mu_2-r, \dots, \mu_r\} + \dots$$

Let $\rho=(1^a 2^b 3^c \dots)$ denote any partition of any integer m . Put $S_\rho = S_1^a S_2^b S_3^c \dots$, and let h_ρ denote the order of the corresponding class of the symmetric group on m symbols. Let $S'_r = S_r + 1$, and $S'_\rho = S_1'^a S_2'^b S_3'^c \dots$.

The author obtains a formula for the Kronecker product of two representations of the symmetric group which may be expressed as

$$(\mu \times \mu') \rightarrow \{\mu\}\{\mu'\} + \sum h_\rho \{\mu/S_\rho\}\{\mu'/S'_\rho\} S'_\rho,$$

summed for all partitions ρ of all integers m . The terms of depth from $p+p'$ to $p+p'-5$ and from $p-p'$ to $p-p'+5$ are given in detail.

D. E. Littlewood (Bangor).

Wang, Shih-Chiang. Representation of ordered abelian groups and ordered rings of finite degree. Acta Math. Sinica 5 (1955), 425-432. (Chinese. English summary)

Let G be a simply ordered abelian group. For any two elements a, b of G , we write $a \sim b$, if there exist positive integers h, k such that $|a| \leq h|b|$, $|b| \leq k|a|$. We write $a \ll b$, if $|a| < |b|$ and if $a \sim b$ is false. G is said to be of degree n , if there exist $n+1$ elements a_i ($0 \leq i \leq n$) of G such that $0 = a_0 \ll a_1 \ll \dots \ll a_n$ and that every element of G is \sim with some a_i . It is proved that every simply ordered Abelian group of degree n is isomorphic with a subgroup of the lexicographically ordered additive group of all n -dimensional real vectors. From this result, the author also derives a representation theorem for simply ordered rings of finite degree.

Ky Fan (Notre Dame, Ind.).

Vinogradov, A. A. On the theory of ordered semigroups. Ivanov. Gos. Ped. Inst. Uč. Zap. Fiz.-Mat. Nauki 4 (1953), 19-21. (Russian)

Mal'cev [Math. Ann. 113 (1937) 686-691] exhibited a (cancellable) semigroup H not embeddable in a group. By

linearly ordering H , the author shows that not every ordered semigroup can be extended to a group. By contrast, Alimov [Izv. Akad. Nauk SSSR. Ser. Mat. 14 (1950), 569-576; MR 12, 480] demonstrated impossibility when the ordered semigroup is commutative.

R. A. Good (College Park, Md.).

Thierrin, Gabriel. Sur la théorie des demi-groupes.

Comment. Math. Helv. 30 (1956), 211-223.

The first part of this paper gives conditions for a semigroup D to be a group, in terms of equivalences on D . Given a set H of elements of D , equivalences R_H , θ_H , and ϕ_H are defined as follows: aR_Hb if and only if, for each x of D , $ax \in H$ if and only if $bx \in H$; $a\theta_Hb$ if and only if, for each x of D , $a \in Hx$ if and only if $b \in Hx$; $a\phi_Hb$ if and only if, for each x of D , $x \in Ha$ if and only if $x \in Hb$. Equivalences R_H , θ_H , and ϕ_H are defined similarly, using left-multiplication.

Then D is a group if and only if its right-congruences and left-congruences are the equivalences of the form θ_H

and θ_H respectively. Also, D is a group if and only if, for each right-congruence R , there is an R -class H such that $R = \phi_H$; and similarly for left-congruences. A third theorem gives conditions for D to contain an element r such that, for each x of D , there exist elements x' and x'' for which $xx' = x''x = r$; and this leads to similar theorems for semi-groups with cancellation.

The second part of the paper is concerned with conditions for D^2 not to have proper ideals. For example, this happens if and only if every proper ideal of D contains D^2 .

The third part analyses semi-groups as unions of disjoint semi-groups. For example, every finite semi-group is a union of disjoint semigroups A each of which has the following property: if a and b are in A , then there are elements a_1, \dots, a_n in A such that $a_1 = a$, $a_n = b$, and $a_i A \cap a_{i+1} A \neq \emptyset$ for each i .

H. A. Thurston.

See also: Behrend, p. 702; Iwasawa, p. 714; Hewitt and Zuckerman, p. 754; Cartier, p. 762; Dieudonné, p. 763; Yokota, p. 774.

THEORY OF NUMBERS

Gardiner, Verna; Lazarus, R.; Metropolis, N.; and Ulam, S.

On certain sequences of integers defined by sieves. Math. Mag. 29 (1956), 117-122.

The authors generate a sequence of numbers

1, 3, 7, 9, 13, 15, 21, 25, 31, 33, 37, 43, ...

called lucky numbers, by the following variant of Eratosthenes' sieve for the primes. Beginning with the natural numbers we delete all those occupying positions divisible by 2, i.e. all even numbers. The remaining odd numbers are now attacked by deleting every third one, that is every term whose rank is a multiple of 3. The result is

1, 3, 7, 9, 13, 15, 19, 21, ...

We now delete every seventh number (beginning with 19) and every ninth element of what remains etc. Lucky numbers are very much like primes in their distribution at least as far as 48600. They show roughly the same densities and the same frequencies of gaps. The number of luckies of the form $4n+1$ less than x is nearly equal to the number of the form $4n+3$, the latter being more numerous up to about $x=27000$. Their distribution into residue classes modulo 5 is also fairly uniform. Every even number less than 100000 is the sum of two lucky numbers.

D. H. Lehmer (Berkeley, Calif.).

Sierpinski, W. Les nombres de Mersenne et de Fermat.

Matematiche, Catania 10 (1955), 80-91.

This is an interesting expository account of recent results and conjectures about the numbers 2^n-1 and $2^{2^n}+1$. Besides the new results of the SWAC on the factorization of these numbers, the problem of when a Mersenne number is triangular is discussed. According to a result of Schinzel and Wakulicz, this question is equivalent to asking when is

$$|2^{(m-1)/2} \csc \theta \sin m\theta| = 1 \quad (\tan \theta = \sqrt{7}).$$

There seem to be but 5 such values of m : 1, 2, 3, 5, 13. The corresponding triangular Mersenne numbers are 0, 1, 3, 15, 4095. While on the subject of perfect numbers the concepts of pseudoperfect and quasiperfect numbers are defined. The former number, like $12=2+4+6$, is the sum of a subset of its proper divisors. Odd pseudoperfects exist without limit, the least being 945. A quasiperfect

number n is the sum of its non-trivial divisors, i.e. all divisors except 1 and n . It may be that no number is quasiperfect.

D. H. Lehmer (Berkeley, Calif.).

Sierpiński, Waclaw. Sur une propriété des nombres naturels. Ann. Mat. Pura Appl. (4) 39 (1955), 69-74.

The following two results are established. (1) Let the decomposition of an integer n into powers of primes be written in the form $n=2^{\alpha_1}p_1^{\alpha_2}\dots p_k^{\alpha_k}$, where p_1, \dots, p_k are distinct odd primes and $p_1 < \dots < p_k$. Then every integer not exceeding n can be represented as the sum of suitable distinct positive divisors of n if and only if the inequalities

$$p_i \leq 1 + \sigma(2^{\alpha_1}p_1^{\alpha_2}\dots p_{i-1}^{\alpha_{i-1}}), \quad i=1, \dots, k,$$

are satisfied. Here $\sigma(m)$ denotes the sum of all positive divisors of m . (2) Let n_1, n_2, \dots, n_k be an increasing sequence of positive integers. Then every integer not exceeding $n_1 + \dots + n_k$ can be represented as a sum of distinct n_i 's if and only if $n_i \leq 1 + n_1 + \dots + n_{i-1}$ ($i=1, \dots, k$).

L. Mirsky (Sheffield).

Schinzel, A.; et Sierpinski, W. Sur l'équation $x^2+y^2+1=xyz$. Matematiche, Catania 10 (1955), 30-36.

It is proved by the method of descent that $x^2+y^2+1=xyz$ has no solutions in integers except when $z=3$, in which case all solutions are given essentially by consecutive pairs x, y of the sequence obtained by deleting the even-numbered terms of the Fibonacci sequence. Applications are given of the theorem and the method of proof. It is shown that $u^2-(x^2-4)v^2=-4$ has solutions only if $x=3$. The set of all pairs x, y such that x divides y^2+1 and y divides x^2+1 is obtained; this result, in more general form, was derived in a different way by W. H. Mills [Pacific J. Math. 3 (1953), 209-220; MR 14, 950]. Finally it is established that for any given $n \geq 2$ there are infinitely many pairs of integers x, y such that x divides y^n+1 and y divides x^n+1 .

I. Niven (Eugene, Ore.).

Selmer, Ernst S. The Diophantine equation $\eta^2=\xi^3-D$. A note on Cassels' method. Math. Scand. 3 (1955), 68-74.

The author has recently set up a "second descent" for the diophantine equation $\eta^2=\xi^3-C\xi-D$ the "first descent" being related to halving the elliptic argument

("On Cassels' conditions for the solubility of the diophantine equation $\eta^3 = \xi^3 - D$ ", to appear in *Archiv für Math. og Naturv.*). Here he gives strong numerical evidence that the number of generators "lost" in the second descent is even, at least when $C=0$ and $-D$ is a perfect square. He uses the known relation between (*) $x^3 + y^3 + Az^3 = 0$ and $\eta^3 = \xi^3 + 24A^2$ together with his extensive numerical information about (*). [For a somewhat analogous "second descent" for (*) (the first descent being related to division of the argument by $\sqrt{(-3)}$) and for a similar conjecture, see Selmer, *Math. Scand.* 2 (1954), 49-54; MR 16, 14.] J. W. S. Cassels (Cambridge, England).

Mills, W. H. Certain Diophantine equations linear in one unknown. *Canad. J. Math.* 8 (1956), 5-12.
Let

$$F(x, y) = \alpha x^2 + \beta xy + \gamma y^2 + \delta x + \epsilon y + \xi$$

where $\alpha, \beta, \gamma, \delta, \epsilon, \xi$ are integers and $\alpha | (\beta, \delta), \gamma | (\beta, \epsilon)$. If x_1, y_1 are integers and $F(x_1, y_1) = 0$, the other solutions x_0, y_0 of $F(x, y) = 0, F(x_1, y) = 0$ are also integers. In this way each integer solution of $F(x, y) = 0$ gives rise, in general, to a sequence of infinitely many. This process goes back at least to A. Hurwitz [*Math. Werke*, Bd. II, Birkhäuser, Basel, 1933, pp. 410-421; ignored by the author] and has been recently exploited by several workers [E. S. Barnes, *J. London Math. Soc.* 28 (1953), 242-244; MR 14, 725; A. and R. Brauer, *Jber. Deutsch. Math. Verein.* 36 (1927), Abt. 2, 90-92; K. Goldberg, M. Newman, E. G. Straus and J. D. Swift, *Arch. Math.* 5 (1954), 12-18; MR 15, 857; W. H. Mills, *Pacific J. Math.* 3 (1953), 209-220; *Proc. Amer. Math. Soc.* 5 (1954), 473-475; MR 14, 950; 16, 13]. The author considers $F(x, y) = N(x, y) - zD(x, y)$, where $N(x, y) = ax^2 + bxy + cy^2 + dx + ey + f$, $D(x, y) = pxy + qx + ry + s$ and z is a variable integer. Here a, \dots, s are integers and

$$a | (b, d, p, q), c | (b, e, p, r).$$

A sequence of solutions is said to be of type I if it contains a solution of $N(x, y) = D(x, y) = 0$; of type II if it contains a solution with $y = -q/p$ or with $x = -r/p$. If sequences types I, II exist, they exist for infinitely many values of z . An infinite-descent argument shows that other sequences of solutions exist for at most a finite number of z . The author specifies those values of z for which infinitely many sequences of solutions exist. He shows that if a sequence is periodic its period is one of 1, 2, 3, 4, 6, 8, 12, all these cases actually occurring.

J. W. S. Cassels (Cambridge, England).

Gupta, Hansraj. Some properties of quadratic residues. *Math. Student* 23 (1955), 105-107.

If S is a set of integers and t an integer, then $t \oplus S$ denotes the set of integers obtained by adding t to each of the elements of S . R denotes the set of quadratic residues modulo p^a and N the set of non-residues modulo p^a . The number of quadratic residues and the number of quadratic non-residues of the following sets are computed: $n \oplus R$ ($n \in N$), $r \oplus N$ ($r \in R$), $r \oplus R$ and $n \oplus N$. H. Bergström.

Carlitz, L. A note on nonsingular forms in a finite field. *Proc. Amer. Math. Soc.* 7 (1956), 27-29.

Theorem 1. Let $(q, m) = 1, r \geq 1$. There exist homogeneous polynomials f_1, \dots, f_r of degree m with coefficients in $GF(q)$, that vanish simultaneously only at $(0, \dots, 0)$ and such that $f = \xi_1 f_1 + \dots + \xi_r f_r$ ($\xi_i \in GF(q)$) has no singular point (except at $(0, \dots, 0)$). Theorem 2.

Let $f_1(y), \dots, f_r(y)$ be arbitrary homogeneous polynomials of degree m with coefficients in $GF(q)$ and let $s > r(m-2)$. Then for arbitrary $y_i \in GF(q)$ there exist $\xi_i \in GF(q)$ such that the Hessian $H_f = \det(\partial^2 f / \partial y_i \partial y_j)$ ($i, j = 1, \dots, r$) vanishes at (y_1, \dots, y_r) . A. L. Whiteman.

Kasch, Friedrich. Abschätzung der Dichte von Summenmengen. *Math. Z.* 62 (1955), 368-387.

Let A, B, \dots denote sets of non-negative integers. The set \bar{A} consists of those non-negative integers which do not belong to A . The set $C = A + B$ consists of those integers x which permit representation $x = a + b, a \in A, b \in B$. Let $A(n), \dots$ denote the number of positive elements $\leq n$ of A . Let OCB, ICB and let $l(m)$ denote the smallest l for which $m = b_1 + \dots + b_l; b_1 \in B, \dots, b_l \in B$ is solvable. Put $\lambda = \sup_{k=1, \dots, n} k^{-1} \sum_{i=1}^k l(m)$. Generalizing Erdős' original method, the author proves

$$(1) \lambda k(C(n) - A(n)) \geq$$

$$\begin{cases} \sum_{i=1}^k A(n-i) - A(n)(A(n)-1)/2 + \Gamma(k) \\ k\bar{A}(n) - \sum_{i=1}^k \bar{A}(m) - \bar{A}(n)(\bar{A}(n)-1)/2 + \bar{\Gamma}(k) \end{cases}$$

($k = 1, 2, \dots, n$). Here

$$\Gamma(k) = \sum_{i=1}^{n-k-1} A(m)(A(m+k+1) - A(m+k))$$

if $k \leq n-2$ and $\Gamma(n-1) = \bar{\Gamma}(n) = 0$; $\Gamma(k)$ is obtained by replacing A by \bar{A} [Erdős, *Acta Arithmetica* 1 (1936), 197-200. Let $0 < \alpha = \inf_{k=1, \dots, n} A(k)/k < 1$. The formulae (1) yield the known estimates of $C(n)/n$ in terms of α and λ as well as the following improvement: $C(n)/n \geq \alpha(1 + c(1-\alpha)\lambda^{-1})$, where

$$c = \max((1 + \alpha^{\frac{1}{2}} + \alpha)(1 + \alpha^{\frac{1}{2}})^{-2}, (1 + (1 - \alpha)^{\frac{1}{2}} + (1 - \alpha)) \times (1 + (1 - \alpha^{\frac{1}{2}})^{-2}).$$

Similar results are obtained from (1) which involve asymptotic densities ((1) remains valid if $A(\bar{A})$ is replaced by $C(\bar{C})$). This leads to estimates of an entirely different structure). P. Scherk (Saskatoon, Sask.).

Brauer, Alfred. On the Schnirelmann density of the sum of two sequences. *Math. Z.* 63 (1956), 529-541.

Let A, B, \dots denote sets of non-negative integers. The set $C = A + B$ consists of those integers c which permit representations $c = a + b, a \in A, b \in B$. Let $A(k), \dots$ denote the number of positive elements $\leq k$ of A, \dots . Let OCB, ICB and let $g(m)$ denote the smallest number of elements of B whose sum equals m , put $\lambda = \sup_{k=1, \dots, n} k^{-1} \sum_{i=1}^k g(m)$. Let $0 < \alpha \leq \inf_{k=1, \dots, n} A(k)/k$. Refining S. Selberg's method [*Arch. Math. Naturvid.* 47 (1944), no. 8, 111-118; MR 8, 566; 13, 1138], the author proves: Let $\alpha \leq \frac{1}{2}$. Then $c(n)/n \geq \alpha + \frac{1}{2}(\lambda^2 - (3\alpha(1-\alpha))^{\frac{1}{2}})$. For $\alpha \leq \frac{1}{2}$, this is an improvement of Selberg's estimate $C(n)/n \geq \alpha + 3\alpha(1-\alpha)/4\lambda$. However, it is not as good as Kasch's [see the paper reviewed above]. P. Scherk (Saskatoon, Sask.).

Salié, Hans. Über die Dichte abundanter Zahlen. *Math. Nachr.* 14 (1955), 39-46.

Denote by $A(x; \lambda)$ the number of positive integers $n \leq x$ such that $\sigma(n)/n \geq \lambda$. Here $\sigma(n)$ denotes the sum of the divisors of n . Davenport, Behrend and S. Chowla proved, independently, the existence of the limit

$$\lim_{x \rightarrow \infty} \frac{A(x; \lambda)}{x} = D(\lambda)$$

for every $\lambda > 1$, and further that $D(\lambda)$ is a continuous function of λ . Sharpening inequalities due to Behrend [S.-B. Preuss. Akad. Wiss. 1932, 322-328; 1933, 280-293], the author proves Satz 1: $D(3/2) > 0.569$; $D(2) > 0.246$; $D(3) > 0.018$. A special case of his Satz 2 says that every integer $n \geq 33426748355$ is either an abundant number or a sum of two such numbers. S. Chowla.

Stöhr, Alfred. Gelöste und ungelöste Fragen über Basen der natürlichen Zahlenreihe. I, II. J. Reine Angew. Math. 194 (1955), 40-65, 111-140.

This is an interesting and stimulating report on the subject of the title. Denote by Z the set of all non-negative integers, by B any subset of Z . For $n \geq 0$ let $B(n)$ denote the number of $b \in B$ with $0 < b \leq n$. Let h be a natural number. hB denotes the set of all numbers which are expressible as a sum of h summands belonging to B . B is called a basis of order h if $hB = Z$. Set

$$\beta_1 = \inf B(n)n^{-1/h}, \quad \beta_2 = \liminf B(n)n^{-1/h}, \\ \beta_3 = \limsup B(n)n^{-1/h}, \quad \beta_4 = \sup B(n)n^{-1/h}.$$

Denote by $v_1(h), v_2(h), v_3(h), v_4(h)$ the lower bounds of $\beta_1, \beta_2, \beta_3, \beta_4$ when B runs through all bases of order h . The author proves

$$v_1(h) = h^{-1/h},$$

and draws attention to the unsolved problems of the estimations of $v_2(h), v_3(h), v_4(h)$.

In part II, the author gives an account of a host of related problems. For example, a set of natural numbers a_1, a_2, a_3, \dots (written in increasing order of magnitude) is a B_2 -sequence of Sidon if $a_i + a_j = a_k + a_l$ has only the trivial solutions. Let $A(n)$ denote the number of $a's \leq n$. Write $\Phi(n) = \max A(n)$, the max being taken over all possible sequences of $a's$, which are B_2 sequences of Sidon. Erdős and Turán conjectured, and Chowla [Proc. Nat. Acad. Sci. India. Sect. A. 14 (1944), 1-2; MR 7, 243] and Erdős [J. London Math. Soc. 19 (1944), 208; MR 7, 242] proved that

$$\lim_{n \rightarrow \infty} \frac{\Phi(n)}{\sqrt{n}} = 1.$$

The author mentions the following recursive construction of an infinite B_2 -sequence of Sidon due to Mian and Chowla [Proc. Nat. Acad. Sci. India Sect. A. 14 (1944), 3-4; MR 7, 243]. Set $a_1 = 1$. Having fixed a_1, \dots, a_m , we define a_{m+1} as the smallest natural number different from all $a_r + a_s - a_t$ with $1 \leq r, s, t \leq m$. We thus get $a_1 = 1, a_2 = 2, a_3 = 4, a_4 = 8, a_5 = 13, a_6 = 21, a_7 = 31, a_8 = 45, a_9 = 66, a_{10} = 81$, etc. Stöhr proves trivially that $a_n \leq n^3$ and raises the problem whether $a_n = O(n^3)$ can be improved. There is reference also to work of A. Brauer, H. Rohrbach, L. Schnirelmann, S. Selberg, A. Stöhr. S. Chowla.

Ankeny, N. C. Quadratic residues. Duke Math. J. 21 (1954), 107-112.

With the consent of the original reviewer [cf. MR 15, 777], the following review should replace the first one. The author attempts to prove that the least positive quadratic nonresidue $n(k)$ with respect to the prime modulus $k \equiv 3 \pmod{4}$ satisfies the inequality

$$(1) \quad n(k) < k^\epsilon$$

for arbitrary $\epsilon > 0$ and $k > k_0(\epsilon)$.

However, the author's reasoning is incorrect. In

applying the relation

$$(2) \quad \lim_{D \rightarrow \infty} \frac{\ln N_{mm}^*}{\ln D} \frac{\max_{\chi} |\sum_{n \leq x} \chi(n)|}{D^{\frac{1}{2}} \ln D} = 0$$

[proved in the paper of Linnik and Rényi, Izv. Akad. Nauk SSSR. Ser. Mat. 11 (1947), 539-546, Th. IB; MR 9, 333]. The author does not notice that N_{mm}^* designates the modulus of the power residue least in absolute value. In the case of interest, $D = k \equiv 3 \pmod{4}$, the relation (2) is trivial since $\chi(-1) = -1$ and $N_{mm}^* = 1$. If (2) were true only for positive nonresidues, then (1) really could be proved, and considerably simpler than in the paper.

K. A. Rodoskii (RŽMat 1955, no. 1079).

Algebraic Number Theory

Cohen, Eckford. A class of arithmetical functions. Proc. Nat. Acad. Sci. U.S.A. 41 (1955), 939-944.

An even function $f(n, r)$ in a field F of characteristic 0, containing the r th roots of unity, is characterized by the property $f(n, r) = f(e, r)$ for $e = (n, r)$. It is proved that the even functions are those functions which can be given in the form $f(n, r) = \sum a(n, r) g(d, r/d)$, where $g(a, b)$ is an arbitrary single-valued function in F of two positive integral variables a, b . The class of these functions is also identical with the class of functions having a finite Fourier representation $f(n, r) = \sum a(r) a(d) c(n, d)$, $a(d) \in F$, with Ramanujan sums $c(n, d)$. Using these results, the author gives a new proof of a theorem of Nicol and Vandiver about the number of solutions of the congruence $n = x_1 + \dots + x_s \pmod{r}$ in x_i , with $(x_i, r) = 1$.

H. Bergström (Göteborg).

Carlitz, L. A note on power residues. Duke Math. J. 22 (1955), 583-587.

If p is a prime $\equiv 1 \pmod{4}$, $h = h(p)$ the class number of the real quadratic field $R(\sqrt{p})$, $\epsilon = (1 + \sqrt{p})/2$ the fundamental unit of the field ($\epsilon > 1$), Ankeny, Artin and Chowla proved [Proc. Nat. Acad. Sci. U.S.A. 37 (1951), 524-525; MR 13, 212]

$$2uh/t \equiv (A+B)/p \pmod{p},$$

where A is the product of the quadratic residues and B the product of the non-residues in the interval 1, $p-1$. Now the right-hand side of the above formula can also be replaced by twice the $\frac{1}{2}(p-1)$ th Bernoullian number (in the even suffix notation). The author extends and generalizes this last result, and obtains a number of interesting formulae. Considerable use is made of results from N. Nielsen's classic "Traité élémentaire des nombres de Bernoulli" [Gauthier-Villars, Paris, 1923]. S. Chowla.

Carlitz, L. Note on the class number of quadratic fields. Duke Math. J. 22 (1955), 589-593.

Ankeny, Artin and Chowla [Ann. of Math. (2) 56 (1952), 479-493; MR 14, 251] obtained an arithmetical expression (in contrast to the classical formula which involves logarithms and infinite series) for the class-number of the real quadratic field $R(\sqrt{p})$ where p is a prime $\equiv 1 \pmod{4}$. The author extends this result to the general case when p is replaced by an arbitrary square-free positive integer. He also points out that some of the results of Ankeny, Artin, Chowla were anticipated by A. A. Kiselev [Dokl. Akad. Nauk SSSR (N.S.) 61 (1948), 777-779; MR 10, 236]. S. Chowla (Boulder, Colo.)

Litver, E. L. A fundamental basis of the composite of quadratic fields. Rostov. Gos. Univ. Uč. Zap. Fiz.-Mat. Fak. 32 (1955), no. 4, 29-36. (Russian)

The author shows that the field K_n composed of n independent quadratic fields $K(a_i^2)$ ($i=1, 2, \dots, n$) over the rational base field has discriminant equal to the product of the discriminants of all 2^n-1 quadratic sub-fields. The author notes that any one a_i can be replaced by $a_i a_j$ ($i \neq j$) without affecting K_n ; therefore the cases can be kept under control by the restriction $a_i \equiv 1 \pmod{4}$ for $i > 2$. Then the basis can be constructed simply and explicitly in terms of the discriminants of these 2^n-1 sub-fields. In concluding, the author notes that the result is valid for any quadratic base field of class number unity.

Harvey Cohn (Detroit, Mich.).

Calloway, J. M. On the discriminant of arbitrary algebraic number fields. Proc. Amer. Math. Soc. 6 (1955), 482-489.

The following general theorem is proved. The absolute value of the discriminant d of an arbitrary algebraic number field K of degree $n \geq 2$ is always greater than $(\pi/3)^{2r_1} (n=r_1+2r_2)$. This result is obtained from the identity

$$|d| = \left(\frac{\pi}{3}\right)^{r_1} + (2\pi)^{r_1} \sum_{(1/\delta) \mid \lambda} \left(\frac{\sin \pi \lambda}{\pi \lambda}\right)^2 \sum_{l=r_1+1}^{r_1+r_2} G(2\pi|\lambda|),$$

where λ runs through all the elements of the ideal $1/\delta$ except 0, δ is the different of K , and

$$G(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m+3)m!(m+2)!}.$$

This identity, which is a generalization of an identity of Siegel is a specialization of a more general identity which is given here.

H. Bergström (Göteborg).

Sukallo, A. A. On determination of the index of a field of algebraic numbers. Rostov. Gos. Univ. Uč. Zap. Fiz.-Mat. Fak. 32 (1955), no. 4, 37-42. (Russian)

If K is an algebraic number field of degree n over the rationals and if Θ is an arbitrary primitive integer in K , the index of Θ is defined as the index of the sub-module having basis $1, \Theta, \dots, \Theta^{n-1}$ in the larger module of all integers $\omega_1, \omega_2, \dots, \omega_n$ in K ; and J , the index of K , is defined as the greatest common divisor of these individual indices. Then J is known to contain as prime factors only those p less than n [Żyliński, Math. Ann. 73 (1913), 273-274] but the value of the exponent of p is generally unknown. [For an elementary treatment of $n=3$ see Tornheim, Pacific J. Math. 5 (1955), 623-631; MR 17, 463.] The author solves the case of those p which decompose into l ($> p$) prime ideals of first degree. Let $p = \prod_1^l P_i$, $e_1 \geq e_2 \geq \dots \geq e_l$, and define P, Q, R by $P = \sum_1^l e_i$, $n = PQ + R$ ($0 \leq R \leq P$). Then p divides J with exact exponent $N = \frac{1}{2}PQ(Q-1) + QR$. In fact, the author constructs the p -adic basis $\mu_0, \mu_1, \dots, \mu_{n-1}$, where $\mu_{HP+k} = \Phi(\Theta)^H \Theta^k / p^H$ ($0 \leq k \leq P$), and where Φ is now specialized as follows: In the indeterminate form $w = \sum_1^n u_i \omega_i$ let the defining polynomial be factored modulo p into $\prod_1^l \Phi_i$ so that (Φ_i, p) corresponds to P_i . Specialize w to Θ by choosing u_i so that P_1, \dots, P_p specialize to p different first-degree polynomials $F_1(\Theta), \dots, F_p(\Theta)$, and so that $F_i(\Theta) \equiv 0 \pmod{P_i}$ and $\not\equiv 0 \pmod{P_i^2}$. Finally set $\Phi(\Theta) = \prod_1^l F_i(\Theta)^{e_i}$. As an application the author considers the field generated by a root of $f(x) = x^5 + 4qa_1x^4 + 2qa_2x^3 + 4qa_3x^2 + 16qa_4x + 128qa_5$, for q prime and a_i odd, with the statement that

$2 = P_1^2 P_2 P_3 P_4$, whence $N=2$ for $p=2$. [Reviewer's remark: Some further qualifications are necessary, e.g., when $q=3, a_1=1$, then $f(x)$ would fail to have the three linear factors modulo 32].

Harvey Cohn.

Kubota, Tomio. A note on units of algebraic number fields. Nagoya Math. J. 9 (1955), 115-118.

Using Grunwald's existence theorem (in the formulation of Hasse), the author proves the following theorem. Let k be an algebraic number field, l a prime number, E_k the group of units of k and H a subgroup of E_k containing all l -th powers of elements of E_k . Assume that, for every $\eta \in H$, $k(\eta^{1/l})$ is always ramified over k whenever k contains an l -th root $\zeta_l (\neq 1)$ of unity. Then there are infinitely many cyclic extensions K/k of degree l with the following properties: a) $N_{K/k} E_k = H$, where E_k is the group of units of K ; b) if an ideal \mathfrak{A} of k is principal in K , then \mathfrak{a} is principal in k .

H. Bergström (Göteborg).

Tôyama, Hiraku. A note on the different of the composed field. Kôdai Math. Sem. Rep. 7 (1955), 43-44.

Let K_1 and K_2 be extensions of finite degree of a field K of algebraic numbers, and K_3 the field generated by K_1 and K_2 ; let D_i ($i=1, 2, 3$) be the different of K_i with respect to K . The author observes that D_3 is a multiple of the L.C.M. D_4 of D_1 and D_2 and divides $D_1 D_2$, but that it may happen that D_3 is distinct from D_4 and $D_1 D_2$.

C. Chevalley (Paris).

Dwork, B. The local structure of the Artin root number. Proc. Nat. Acad. Sci. U.S.A. 41 (1955), 754-756.

Let k be a field of algebraic numbers, K/k a normal extension of finite degree of k , X a character of the Galois group $G(K/k)$ and $L(s, X, K/k)$ the corresponding Artin L -series. Then the functional equation of $L(s, X, K/k)$ involves a certain "root number" $W(X)$. As a function of X , $W(X)$ satisfies certain functional equations, viz. the linearity, factor-group and induced character properties. If one considers only linear characters X , then $W(X)$ has a product formula which expresses it in terms of factors $W_p(X_p)$ attached to the various prime divisors p of k , X_p being the character of the local Galois group at p defined by X and W_p being a function on the linear characters of local Galois groups. The author shows that, in the general case, there is a formula of the type $W(X) = \pm \prod_p W_p(X_p)$, where W_p is a function on the characters of local Galois groups at p which has the linearity, factor group and induced character properties; the \pm may be omitted in case K/k is of odd degree. The proof, which is only briefly sketched, depends on two theorems: one gives a criterion for a function on linear characters of the groups in a certain family Δ of nilpotent groups to be extendable to a function on all characters of the groups in Δ with the linearity, factor group and induced character properties; the second one gives a criterion of extendability of a function but this time to all characters of local Galois groups, but under an assumption which guarantees extendability of a certain product function (over all prime divisors) to characters of global Galois groups.

C. Chevalley (Paris).

Iwasawa, Kenkichi. On Galois groups of local fields. Trans. Amer. Math. Soc. 80 (1955), 448-469.

Let p be a prime number, \mathbb{Q}_p the field of p -adic numbers, Ω its algebraic closure and k a finite extension of \mathbb{Q}_p in Ω . The object of the paper is to study the structure of the Galois group $G(\Omega/k)$ of Ω/k (for any Galois extension

K/L , $G(K/L)$ will denote the Galois group of this extension). The author introduces the intermediary field V , largest tamely ramified extension of k in Ω . The problem is then decomposed into three parts: 1) to find the group $G(\Omega/V)$; 2) to find the group $G(V/k)$; 3) to determine the extension of $G(\Omega/V)$ by $G(V/k)$ given by the group $G(\Omega/k)$. Problems 1) and 2) are completely solved, problem 3) partially.

The group $G(\Omega/V)$ may be described as follows. Let F be a free group on countably many generators; then $G(\Omega/V)$ is the projective limit ϕ of the system of all finite factor groups of F which are p -groups. It is obvious that $G(\Omega/V)$ is a factor group of the so defined ϕ ; the reason why it is ϕ itself is that every Galois 2-cocycle of a finite extension of k splits over V , because V contains the maximal unramified extension of k .

The group $G(V/k)$ is described as follows. Let A be the group with two generators a and b and a unique relation $aba^{-1} = bq$, where q is the number of elements of the residue field of k . Then $G(V/k)$ is isomorphic to the projective limit of all finite factor groups of A . This is easily established by observing that V is generated by all roots of unity of orders prime to p and by all e -th roots of an element of order 1 in k , e running over the integers prime to p .

As to problem 3), it is first established that the extension of $G(\Omega/V)$ by $G(V/k)$ splits; this result depends only on the structure of the groups $G(\Omega/V)$ and $G(V/k)$ described above. Let then H be a subgroup of $G(\Omega/k)$ such that $G(\Omega/k) = HG(\Omega/V)$ (it is proved that H may be taken to be compact); it would remain to determine the manner in which H operates on the group $G(\Omega/V)$. This seems to be a difficult problem, especially in view of the fact that there is apparently no canonical way of selecting H . However, the operation of an element s of $G(\Omega/k)$ on the factor group $G(\Omega/V)/G'(\Omega/V) = G(V'/V)$ (where $G'(\Omega/V)$ is the derived group of $G(\Omega/V)$ and V' the maximal abelian extension of V) depends only on the coset of s modulo $G(\Omega/V)$; thus the group $G(V'/V)$ operates on $G(V'/V)$. Making use of class field theory, the problem of finding how $G(V'/V)$ operates on $G(V'/V)$ is substantially reduced to the problem of finding the way $G(E/k)$ operates on the multiplicative group of elements of E when E/k is a finite Galois extension which is tamely ramified. The author solves this problem in a completely explicit manner under certain assumptions on E , these assumptions being such that V is the union of the fields E which satisfy them. These results complete the more general but less explicit results of M. Krasner on the same question [Acta Arith. 3 (1939), 133-173]. Using this, the author gives an explicit description of the way in which $G(V'/V)$ operates on $G(V'/V)$, which it would be a little too long to reproduce here.

C. Chevalley (Paris).

See also: Moriya, p. 578; Ono, p. 582.

Geometry of Numbers, Diophantine Approximation

Cassels, J. W. S. Simultaneous diophantine approximation. II. Proc. London Math. Soc. (3) 5 (1955), 435-448.

[For part I see J. London Math. Soc. 30 (1955), 119-121; MR 16, 574.] It is proved that for every integer $N \geq 1$ there are continuum-many N -tuples of real numbers $\alpha_1, \dots, \alpha_N$ such that

$$r^{1/N} \max |r\alpha_n - p_n| \geq \gamma_N \quad (1 \leq n \leq N)$$

for all integers $r > 0$, p_1, \dots, p_N , where $\gamma_N > 0$ depends only on N . The case $N=2$ was proved, by a different method, already by Davenport [Mathematika 1 (1954), 51-72; MR 16, 223]. J. F. Koksma (Amsterdam).

Barnes, E. S. On linear inhomogeneous Diophantine approximation. J. London Math. Soc. 31 (1956), 73-79.

Let ϕ be irrational and α such that $\phi x + y + \alpha \neq 0$ for any integral x, y . It is then proved that for any k with $0 \leq k \leq \frac{1}{2}$ there exist continuum-many values of ϕ to each of which correspond continuum-many values of α , such that

$$\liminf_{|x| \rightarrow \infty} |x(\phi x + y + \alpha)| = k.$$

It is further proved that the theorem remains true, if in the final statement one replaces $|x| \rightarrow \infty$ by $x \rightarrow \infty$. These theorems include as special cases results of Cassels (unpublished) and S. Fukasawa [Jap. J. Math. 3 (1926), 91-106]. They are proved by the aid of an algorithm used in former cases by the author [Acta Math. 92 (1954), 235-264; MR 16, 802] and by the author and H. P. F. Swinnerton-Dyer [ibid. 92 (1954), 199-234; MR 16, 802]. J. F. Koksma (Amsterdam).

Mordell, L. J. The minimum of an inhomogeneous quadratic polynomial in n variables. Math. Z. 63 (1956), 525-528.

Let $f(X) = \sum a_{rs} x_r x_s + 2 \sum a_r x_r$, with real coefficients, $a_{rs} = a_{sr}$, $a_{rr} > 0$, $\det(a_{rs}) \neq 0$. Then there exist numbers $X = (x_r)$ with assigned residues modulo 1 such that $f(X) \leq k$ if k is given by $\det \begin{pmatrix} A & B \\ C & -k \end{pmatrix} = 0$, where $A = (a_{rs})$, B is the column vector $(\frac{1}{2} a_{rr} + a_r)$, and C the row vector $(a_s - \frac{1}{2} a_{ss})$ and if, further, the indices $1, \dots, n$ can be divided into two sets λ, λ', \dots and μ, μ', \dots such that $a_{\lambda\lambda'} \leq 0$ ($\lambda \neq \lambda'$) and $a_{\mu\mu'} \leq 0$ ($\mu \neq \mu'$). This generalizes a result of Dirichlet for $n=2$. The proof is simple and uses algebraic identities. For $n=2$ the result is shown to be best possible; for $n > 2$ it is pointed out that it cannot be.

L. Tornheim (Berkeley, Calif.).

Rogers, C. A. Mean values over the space of lattices. Acta Math. 94 (1955), 249-287.

Let F denote the fundamental region for real $n \times n$ matrices of determinant 1 defined by Minkowski in his theory of the reduction of quadratic forms. Siegel [Ann. of Math. (2) 46 (1945), 340-347; MR 6, 257] by use of the Lebesgue measure of n^2 -space defined a measure $\mu(\Omega)$ for the matrices Ω of F which is normalized so that $\int_F d\mu(\Omega) = 1$. For a lattice Λ of determinant 1, let Ω be the uniquely determined matrix of F for which $\Lambda = \Omega \Lambda_0$, Λ_0 being the lattice of all points with integral coordinates. If $\varrho(\Lambda) = \varrho(\Omega \Lambda_0)$ be a function defined for all lattices Λ of determinant 1 which is Borel measurable over F

$$\int_F \varrho(\Omega \Lambda_0) d\mu(\Omega)$$

is defined to be the Siegel mean of $\varrho(\Lambda)$. If $\Lambda(\theta_1, \dots, \theta_{n-1})$ be the lattice generated by the points $(\omega, \dots, \theta_1 \omega^{-n+1}), \dots, (0, \dots, \omega, \theta_{n-1} \omega^{-n+1}), (0, \dots, 0, \omega^{-n+1})$, then

$$\lim_{\omega \rightarrow 0} \int_0^1 \dots \int_0^1 \varrho[\Lambda(\theta_1, \dots, \theta_{n-1})] d\theta_1 \dots d\theta_{n-1}$$

is designated by $M_\Lambda[\varrho(\Lambda)]$.

The paper consists of a number of results dealing with $M_\Lambda[\varrho(\Lambda)]$. Conditions are given so that $M_\Lambda[\varrho(\Lambda)] =$

$\int_F \varrho(\Omega \Lambda_0) d\mu(\Omega)$. Among the other results are the following. If Y is a point of the m -space defined by the m n -vectors X_1, \dots, X_m , $0 < m \leq n-1$, and $\varrho(Y) = \varrho(X_1, \dots, X_m)$ is a function for which $\int \varrho(Y) dY$ exists, let

$$\varrho(\Lambda) = \sum \varrho(X_1, \dots, X_m),$$

where X_1, \dots, X_m run through all lattice points of Λ subject to certain auxiliary conditions. It is shown that $M_\Lambda[\varrho(\Lambda)] = K \int \varrho(Y) dY$, where K is determined in terms of the auxiliary conditions.

A combination of the above two results leads to a proof that

$$\begin{aligned} \int_F \sum \varrho(X_1, X_2) d\mu(\Omega) &= \frac{1}{\zeta(n)^2} \iint \varrho(X_1, X_2) dX_1 dX_2 \\ &+ \frac{1}{\zeta(n)} \int \varrho(X_1, X_1) dX_1 + \frac{1}{\zeta(n)} \int \varrho(X_1, -X_1) dX_1, \end{aligned}$$

where the n -vectors X_1, X_2 of the sum run through all the primitive points of the lattice $\Omega \Lambda_0$. The proofs

consisting mostly of elementary manipulations of the integrals are too detailed to be described here.

D. Derry (Vancouver, B.C.).

O'Meara, O. T. Witt's theorem and the isometry of lattices. *Proc. Amer. Math. Soc.* 7 (1956), 9-22.

If $F+G$ and $F+H$ are equivalent quadratic forms in a field, Witt has shown that G and H are field equivalent. This is also true over any local field in which 2 is a unit. The author [*Amer. J. Math.* 77 (1955), 87-116; MR 16, 680] has extended this to any local field in which 2 is a prime and also to the ramified case when F is restricted to be a form in only one variable. In this paper he removes this limitation on F . Also he shows that G and H are equivalent provided that F is the orthogonal sum of totally isotropic binary forms, that is, forms of the type

$$\begin{pmatrix} 0 & \pi^{\lambda(i)} \\ \pi^{\lambda(i)} & 0 \end{pmatrix}$$

where $\lambda(i)$ are arbitrary integers and π is a prime element in the field.

B. W. Jones (Boulder, Colo.).

ANALYSIS

Rodov, A. M. Sufficient conditions for existence of a function of a real variable with given upper bounds of the moduli of the function and its first five derivatives. *Belorussk. Gos. Univ. Uč. Zap. Ser. Fiz.-Mat.* 19 (1954), 65-72. (Russian)

Let M_k be the least upper bound of $|f^{(k)}(x)|$ on $(-\infty, \infty)$. Given a finite set of numbers a_1, \dots, a_n , one asks for conditions which are necessary and sufficient for the existence of a function having $M_{a_j} = a_{a_j}$ ($j=1, \dots, k$). The author has solved the problem for various selections from a_0, \dots, a_4 [*Izv. Akad. Nauk SSSR. Ser. Mat.* 10 (1946), 257-270 = *Amer. Math. Soc. Transl. no. 14* (1950); MR 8, 65; 11, 504]. Here he solves it for (a_0, a_1, a_2, a_3) and for $(a_0, a_1, a_2, a_3, a_4, a_5)$. R. P. Boas, Jr. (Evanston, Ill.).

Lalagué, Pierre. Sur certaines classes de fonctions indéfiniment dérivables. *Ann. Sci. Ecole Norm. Sup.* (3) 72 (1955), 237-298.

[The main results of this thesis were announced in C.R. Acad. Sci. Paris 236 (1953), 2473-2475; 240 (1955), 1041-1043; MR 15, 107; 16, 804.] The author studies classes of functions defined by bounds on their successive derivatives. In the first chapter the functions belong to $C\{M_n\}$, i.e. $|f^{(n)}(x)| \leq C M_n$, and in addition are almost periodic with spectrum contained in a given set. The author introduces a new process of regularization that is appropriate for such functions and uses it to characterize those subclasses that can be continued analytically into a strip, that contain the derivatives of their members, or have the quasi-analytic property. He also discusses similar classes defined by $|f^{(n)}(x)| \leq k M_n$. In the second chapter he solves the problem of equivalence of classes defined on $(0, \infty)$ by such inequalities as $\int_0^\infty e^{2x} |f^{(n)}(x)|^2 dx \leq C M_n$, and similar classes on $(-\infty, \infty)$ (involving e^{x^2}). In the third chapter he determines the smallest class to which $f(\cos \theta)$ must belong when $f(x)$ belongs to a class $C\{M_n\}$ on $[-1, 1]$, and the smallest class to which $f(x)$ must belong when $f(\cos \theta)$ belongs to $C\{M_n\}$ on $[-1, 1]$ or $(-1, 1)$.

R. P. Boas, Jr. (Evanston, Ill.).

Pastidès, Nicolas. Sur la régularisation des suites et des fonctions de M. S. Mandelbrojt. *Rend. Circ. Mat. Palermo* (2) 4 (1955), 132-204.

In the first part of this thesis the author studies functions whose successive derivatives have bounds M_n in an expanding or contracting sequence of intervals. He deduces inequalities for the derivatives in terms of certain regularizations of the sequence $\{M_n\}$, and shows that they cannot be improved. A regularized sequence is defined as the ordinate at abscissa n of a certain discontinuous curve made up of convex polygonal pieces. In the second part the author makes a detailed study of these curves and shows their relevance to the problem of regularizing functions [à la Mandelbrojt, *La régularisation des fonctions*, *Actualités Sci. Ind.*, no. 733, Hermann, Paris, 1938] by proving that the regularization of a given $f(x)$ is the highest curve of the class lying below the graph of $f(x)$.

R. P. Boas, Jr. (Evanston, Ill.).

Toscano, Letterio. Nuove formule sugli operatori permutabili di secondo ordine. *Matematiche, Catania* 10 (1955), 37-43.

If A and X satisfy $AX - XA = 1$, the author writes the relation $(AX^2 - uA)^n = (AX^{1-u})^n X^{un} (X^{1-u}A)^n$, where u is an integer, and n a positive integer. He uses this relation with $A = d/dx$, $X = x$ or $A = -x$, $X = -d/dx$ to obtain identities for functions defined as n th derivatives, notably Laguerre and Jacobi polynomials.

A. Erdélyi.

★ **Smirnov, W. I.** *Lehrgang der höheren Mathematik. Teil II.* Deutscher Verlag der Wissenschaften, Berlin, 1955. xii+580 pp. DM 29.50.

Translation by K. Krienes of vol. 2 of V. I. Smirnov's *Kurs vyssheĭ matematiki* [12th ed., Gostehizdat, Moscow, 1953]. Contents: I) Ordinary differential equations; II) Linear differential equations; III) Multiple and curvilinear integrals; IV) Vector analysis and theory of fields; V) Elements of differential geometry; VI) Fourier series; VIII) Partial differential equations of physics.

Theory of Sets, Functions of Real Variables

Sierpinski, W. L'axiome du choix pour les ensembles finis. *Matematiche, Catania* 10 (1955), 92-99.

Continuing the work of Mostowski [*Fund. Math.* 33

(1945), 137-168; MR 8, 3] and Szmielew [ibid. 34 (1947), 75-80; MR 9, 222], the author studies implications of the form $[n] \rightarrow [m]$. Here $[n]$ is the assertion that corresponding to every class Z of sets consisting of exactly n elements there exists a function f on Z such that $f(A) \in A$ for every A in Z . The deductions are carried out in, say, Zermelo's axiomatic set theory (without, of course, the axiom of choice). Typical result: if $[p]$ is true for every prime $p \leq n$, then $[k]$ is true whenever k is composite and $k \leq 2n+1$.
P. R. Halmos (Chicago, Ill.).

Klůvák, Igor. On systems of sets closed with respect to certain set operations. Mat.-Fyz. Časopis. Slovensk. Akad. Vied 5 (1955), 191-211. (Slovak)
Expository article. E. Hewitt (Princeton, N.J.).

Ōtani, Yu. S. The theory of operations over sets. Uspehi Mat. Nauk (N.S.) 10 (1955), no. 3(65), 71-128. (Russian)

This is an exposition of the general theory of operations over sequences and families of abstract sets. The first three chapters contain the basic definitions and results, which are applied in the last chapter to δ -operations over sets lying in metric spaces. The article may thus be regarded as an extension of the series of articles by Stegol'kov [Uspehi Mat. Nauk (N.S.) 5 (1950), no. 5(39), 14-44; MR 12, 597], Arsenin and Lyapunov [ibid. 5 (1950), no. 5(39), 45-108; MR 12, 597], and Lyapunov [ibid. 5 (1950), no. 5(39), 109-119; MR 12, 597] on the descriptive theory of sets.
F. Bagemihl.

Viola, Tullio. Sull'operazione di passaggio a limite, secondo Borel e de la Vallée Poussin, nella teoria degli insiemi astratti. J. Math. Pures Appl. (9) 35 (1956), 55-65.

Let $\{X\}$ be a collection of non-empty sets which are directed, $>$, by inclusion C . Let Γ be any set and $E(x) \subset \Gamma$ for $x \in X$, $X \in \{X\}$. Define

$$S(X) = \bigcup_{x \in X} E(x), \quad P(X) = \bigcap_{x \in X} E(x)$$

and

$$K'' = \lim_{(X)} E(x) = \bigcap_{X \in \{X\}} S(X), \quad K' = \lim_{(X)} E(x) = \bigcup_{X \in \{X\}} P(X).$$

This generalizes the ordinary \limsup and \liminf of a sequence of sets. Then if $E(x, y)$ is a "generalized double sequence" the author gives various conditions for inclusion relations on iterated limits.
M. E. Shanks.

Denjoy, Arnaud. Approximation des courbes rectifiables par des polygones et intégration. C. R. Acad. Sci. Paris 242 (1956), 850-854.

Given any two point-sets $e, e' \subset E_2$, by distance of e and e' is meant as usual the number $\text{dis}(e, e') = \inf(M, M')$ for all points $M \in e, M' \in e'$; by "écart" of e on e' is meant the number $\text{ec}(e/e') = \sup[\text{dis}(e, M'), M' \in e']$; by mutual "écart" of e and e' , or $\text{ec}(e, e')$ is meant the larger of the écarts of e on e' and of e' on e . The author proves the following theorem. 1) Given a closed rectifiable Jordan curve C in the x -plane E_2 , then in each of the two complementary components C^+, C^- of C there exists a sequence of simple closed polygonal lines P whose Jordan lengths approach the Jordan length of C and whose mutual écarts from C approach zero. The brief proof is based on Vitali's covering theorem and on reasonings of the author's proof of the Jordan theorem [C.R. Acad. Sci. Paris 167 (1918), 389-

391]. It is shown to yield a proof of the general form of the Cauchy-Goursat theorem on functions $f(z)$ holomorphic in C^- and continuous in $C+C^-$ [for another recent proof see J. H. Michael, J. London Math. Soc. 30 (1955), 1-11; MR 16, 577]. Also, a limit theorem for line integrals is given.
L. Cesari (Lafayette, Ind.).

Grošev, A. V. On the degree of uniformity of the distribution of a point set on an interval. Ural. Politehn. Inst. Trudy 51 (1954), 82-85. (Russian)

The author proposes to use as a measure of the uniformity of the distribution of a point set E of measure m on an interval $[0, l]$ the ratio $k = S/\bar{S}$, where

$$S = \int_0^l |\varphi(x) - \varphi_0(x)| dx,$$

$\varphi(x)$ is the measure of that part of E in $[0, x]$, $\varphi_0(x) = mx/l$, and \bar{S} is the maximum of S for m and l fixed. It is proved that $\bar{S} = m(l-m)/2$.
M. M. Day.

Graves, Lawrence M. The theory of functions of real variables. 2d ed. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1956. xii+375 pp. \$7.50.

Aside from minor changes, chapters 1-12 remain as in the first edition [1946; MR 8, 319]. Two new chapters have been added for this edition: ch. 13, The theory of sets and transfinite numbers; ch. 14, Metric spaces.

Geht, B. I. On non-uniform convergence of a sequence of continuous functions. Novočerkassk. Politehn. Inst. Trudy 23 (1953), 161-162. (Russian)

Let E be a closed subset of $[0, 1]$, and write $[0, 1] - E = \bigcup_{k=1}^{\infty} (a_k, b_k)$ with $a_k, b_k \in E$ and $(a_k, b_k) \neq (a_j, b_j)$ for $k \neq j$. Let $\{a_{kn}\}_{n=1}^{\infty} \subset (a_k, b_k)$ and $\{b_{kn}\}_{n=1}^{\infty} \subset (a_k, b_k)$ have limits a_k and b_k , respectively. If $f_n(x)$ is one at $(a_k + a_{kn})/2$ and $(b_k + b_{kn})/2$ ($k=1, 2, \dots, n$), zero on E , (a_{kn}, b_{kn}) ($k=1, 2, \dots, n$) and $[0, 1] - \bigcup_{k=1}^n (a_k, b_k)$, and linear elsewhere on $[0, 1]$, then $f_n(x) \rightarrow 0$. If E is nowhere dense, then the set of points of non-uniform convergence of $\{f_n\}_1^{\infty}$ [Book reviewed above, p. 99] is precisely E .
A. E. Livingston (Seattle, Wash.).

Levšunov, M. T. On the representation of functions by means of the limit sign. Stavropol. Gos. Ped. Inst. Sb. Nauč. Trud. 1952, no. 8, 129-134 (1953). (Russian)

The example $\lim_m \lim_n \cos^{2^n}(m! \pi x)$ is discussed to point out that the limit function of a double sequence of continuous functions can be discontinuous everywhere.
A. E. Livingston (Seattle, Wash.).

Klee, V. L., Jr. Solution of a problem of E. M. Wright on convex functions. Amer. Math. Monthly 63 (1956), 106-107.

The author constructs a function f that is convex in the sense that $f(\frac{1}{2}x + \frac{1}{2}y) \leq \frac{1}{2}f(x) + \frac{1}{2}f(y)$ for all real x, y , but not in the sense that $f(x+\delta) - f(x) \geq f(y+\delta) - f(y)$ for all $\delta > 0$ and $x > y$. This solves a problem proposed by E. M. Wright [same Monthly 61 (1954), 620-622].
F. F. Bonsall (Newcastle).

Kenyon, Hewitt. Note on convex functions. Amer. Math. Monthly 63 (1956), 107.

The author gives, independently, a somewhat simpler construction of a function with the property described in the preceding review.
F. F. Bonsall (Newcastle).

Grošev, A. V. On a functional space. Ural. Politehn. Inst. Trudy 51 (1954), 77-81. (Russian)

This note contains some relations between different kinds of convergence of uniformly bounded sequences of non-decreasing functions. *M. M. Day* (Urbana, Ill.).

Sibirskii, K. S. Spaces of measurable functions. Kišinev. Gos. Univ. Uč. Zap. 11 (1954), 49-53. (Russian)

Let f be a non-decreasing function defined on non-negative reals so that $f(0)=0$ and $f(a+b) \geq f(a)+f(b)$ for all $a, b \geq 0$. For each measurable function x defined on $[0, 1]$ let E_x be the set of numbers $\varepsilon \geq 0$ for which the measure of the set where $|x| > \varepsilon$ is not greater than ε . It is shown that $\inf E_x \in E_x$ if E_x is not empty. Let X be the set of x for which E_x is non-empty; then it is verified that X is metrized by setting $\rho_0(x, y) = \text{distance from } x \text{ to } y$. When $f=0$, X is the space M of bounded measurable functions with essential sup norm: when $f(e)=e$, then X becomes the space S of all measurable functions, and the metric is equivalent to the usual one which defines convergence in measure. In general for $f \neq 0$, X is of finite radius and convergence in it is convergence in measure. *M. M. Day* (Urbana, Ill.).

Rodov, A. M. On a criterion of divergence of improper integrals. Belorussk. Gos. Univ. Uč. Zap. Ser. Fiz.-Mat. 19 (1954), 73-74. (Russian)

The author states that if $f(x)$ is continuous on $[1, \infty]$ and $\int_1^\infty f(x) dx$ converges then $\int_1^\infty x^{-2} f(x) dx$ diverges. His proof uses the calculus of variations. [When $f(x) \geq 0$, the theorem is more easily proved by applying Schwarz's inequality to $\int \{x^{-1} f(x) - 1\} f(x) dx$. If $f(x)$ is allowed to change sign, the theorem is false.] *R. P. Boas, Jr.*

Vainberg, M. M. On a form of the (C)-property of functions. Moskov. Oblast. Pedagog. Inst. Uč. Zap. Trudy Kafedr. Mat. 21 (1954), 65-72. (Russian)

Let B be a measurable set in some Euclidean space E^n , $g(x, u)$ a real function defined for $x \in B$ and $u \in E^n$. The author proves that g is continuous in u for almost all $x \in B$ and measurable in x for each $u \in E^n$ if and only if for every $\eta > 0$ there exists a closed set $F \subset B$, with $\text{mes}(B-F) < \eta$, such that g is continuous on $F \times E^n$. In this case if $u(x)$ is finite almost everywhere and measurable on B , $Hu = g(x, u(x))$ represents an asymptotically continuous operator on the class of functions $u(x)$. *M. Golomb.*

Kropotov, L. L. Representation of the integral formulas of Newton-Leibnitz, Green, Stokes, Gauss and Ostrogradskii by a single formula. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. 13 (1954), 135-151. (Russian)

Expository article. *L. C. Young* (Madison, Wis.).

Marcus, S. Les conditions (T) de Banach pour les fonctions de deux variables. Rev. Univ. "C. I. Parhon" Politehn. București. Ser. Ști. Nat. 4 (1955), no. 8, 15-22. (Romanian. Russian and French summaries)

The author observes that a theorem of Kronrod and Velskii [Kronrod, Uspehi Mat. Nauk (N.S.) 5 (1950) no. 1(35), 24-134; MR 11, 648] remains valid with the slightly weaker hypothesis: " $f(x, y)$, defined in the unit square, satisfies on almost all parallels to the axes the condition T_1 of Banach"; and that its conclusion, which states that "for almost all c the components of the set $f(x, y) = c$ are locally connected," may be thought of as one of the various two-dimensional analogues of T_1 . Another theorem of Kronrod is similarly modified and some corollaries and variants are mentioned. *L. C. Young.*

★ **Vituškin, A. G.** O mnogomernykh variatsiyah. [On multidimensional variations.] Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1955. 220 pp. 5.85 rubles.

The central theme of this tract, a theme with which students of convex figures have long been familiar, is the desirability of attaching to a subset of Euclidean n -space, and also to a function of n real variables, a string of $n+1$ quantities v_k ($k=0, 1, \dots, n$), whose values provide suitably independent items of useful information about the set, or about the function. The classical notions of k -dimensional measure of a set, and of k -dimensional total variation of a function, are inadequate in this respect: for instance if the k -dimensional measure is finite and not zero for one value of k , it is automatically 0 for all larger k and ∞ for all smaller k . One rather more satisfactory string of v_k , which the author does not mention, and which are still related by inequalities, is furnished by the cross-section integrals (Quermassintegrale) of the theory of convex figures; the latter have been extended to sets by Ohmann [Math. Ann. 124 (1952), 265-276; 127 (1954), 1-7; MR 13, 864; 15, 738]. The actual v_k selected by the author are somewhat similar in appearance to the cross-section integrals, but have, for $k=0, 1, \dots, n$, values which are quite independent of one another; v_k is termed k th variation of the set or function concerned. For a function of two variables v_k reduces, when $k=1, 2$, to notions of linear variation and of plane variation due to Kronrod [Uspehi Mat. Nauk (N.S.) 5 (1950), no. 1(35), 24-134; MR 11, 648]. In the case of a set, the k th variation is not in general additive; for instance, when $k=0$ it reduces to the number of components of the set.

The first four chapters treat the definitions and main properties of the k th variations, such as their independence and their relation to classical notions. Chapters V and VI deal, respectively, with some necessary and some sufficient conditions for finiteness of the k th variations of a function for each k , and include analogues of well-known classical results: existence of differential almost everywhere, convergence of Fourier series. Finally Chapter VII is mainly concerned with applications already announced in two notes [Doklady Akad. Nauk SSSR (N.S.) 95 (1954), 433-434, 701-704; MR 15, 945] and in particular with Hilbert's thirteenth problem. The author mentions also the problem of extending his definitions to an arbitrary metric space.

In regard to the main theme of the tract, the particular definitions of the v_k adopted by the author are natural and interesting, and they are reasonably motivated by their applications. Nevertheless, the reviewer does not feel that they supersede in any way classical notions of the theory of several real variables: indeed, to replace the k -dimensional measure of a set for $k=0, \dots, n$ by $n+1$ independent quantities v_k can add but little information unless we make the most unwelcome restriction of generality that all v_k are finite. *L. C. Young.*

Császár, Ákos. Sur une généralisation de la notion de dérivée. Acta Sci. Math. Szeged 16 (1955), 137-159.

Let \mathfrak{R} be an hereditary σ -additive family of sets in R^1 (i.e., if $A \in \mathfrak{R}$ and BCA , then $B \in \mathfrak{R}$, and if each of a sequence of sets $A_n \in \mathfrak{R}$, then $\bigcup A_n \in \mathfrak{R}$). Four derivatives (upper and lower, right and left) of a function $f(x)$ may be defined by taking limits ignoring sets which belong to \mathfrak{R} . The author, by generalising theorems on contingents of plane sets due to Kolmogoroff and Verčenko, Roger Saks and Haslam-Jones [see Saks, Theory of the integral, 2nd ed., Warsaw-Lvov, 1937, pp. 264, 271 and references

there given], shows that the possible distributions of these derivatives are the same as those established by Denjoy for ordinary derivatives, but the exceptional set (which in Denjoy's theorem is a set of Lebesgue measure zero) is in this case the union of a set of measure zero and a set belonging to \mathfrak{N} . He further defines classes of functions $(VB_{\mathfrak{N}})$ and $(VBG_{\mathfrak{N}})$ and studies their differential properties. The paper concludes with a similar generalisation of theorems concerning contingents in R^3 and derivatives in R^3 .
U. S. Haslam-Jones (Oxford).

Hayes, C. A., Jr.; and Pauc, C. Y. Full individual and class differentiation theorems in their relations to halo and Vitali properties. *Canad. J. Math.* 7 (1955), 221-274.

The strong Vitali property implies, according to R. de Possel [*J. Math. Pures Appl.* (9) 15 (1936), 391-409], the "full differentiation theorem" for integrals, i.e. the existence almost everywhere of the derivative of an integral and the equality of this derivative with a Radon-Nikodým integrand, while the weak Vitali property is equivalent to the density property or the "full differentiability" of Lipschitzian integrals. First the Vitali properties and these differentiation theorems are studied here in a very general setting, namely with reference to a general "derivation basis \tilde{B} "; that is the following: Let R be the fundamental set, let \mathbf{M} denote a Boolean σ -algebra of subsets of R with R as unit, let μ be a σ -finite measure defined on \mathbf{M} , and let E be a fixed subset of R . Then it is assumed that to each point x of E there correspond Moore-Smith sequences $M_i(x)$ of \mathbf{M} -sets of finite positive measure, called "constituents", which are said to "converge" to x . Moreover, it is assumed that every (cofinal) subsequence of an x -converging sequence itself converges to x ("heredity axiom"). Then the family of the sequences $M_i(x)$ is the "derivation basis \tilde{B} ". Besides the Vitali properties, two modifications of the weak Vitali property, due to C. A. Hayes [*Proc. Amer. Math. Soc.* 3 (1952), 283-296; *Proc. Cambridge Philos. Soc.* 48 (1952), 374-382; *MR* 14, 28, 29], are also considered.

After establishing the differentiation theorems, the following converse problem is discussed: Do "full differentiation properties" for totally additive set functions imply covering properties of Vitali types? The answer is affirmative. A non-negative totally additive real-valued function defined on \mathbf{M} is called an \mathbf{M} -measure. A μ -integral $\varphi(M) = \int_M f(x) d\mu$ which is finite on the \mathbf{M} -sets of finite measure μ is called a " μ -finite μ -integral". Then, e.g., the following theorem is proved: The Vitali property for μ -finite μ -integrals (or \mathbf{M} -measures) is equivalent to the \tilde{B} -differentiability of every μ -finite μ -integral (or \mathbf{M} -measure, respectively). Relations to a paper of B. Younouch [C. R. (Dokl.) Acad. Sci. URSS (N.S.) 30 (1941), 112-114; *MR* 2, 353] are also discussed; one (unproved) assertion of Younouch remains unsettled.

According to H. Busemann and W. Feller [*Fund. Math.* 22 (1934), 226-256] who considered special Euclidean bases, a weak "halo property" is equivalent to the density property. In a more general setting, weak and strong halo properties were considered by O. Haupt and C. Y. Pauc [*S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss.* 1950, 187-207 (1951); *MR* 14, 544]. In the present paper, halo properties in connection with differentiation or Vitali properties are discussed in a very general manner. Proofs of some fundamental theorems in this respect were already given by C. Y. Pauc [*J. Reine Angew. Math.* 191 (1953), 69-91;

MR 15, 109]. Here some other theorems of such a type are proved, which show that the pointwise halo condition implies the Vitali property for integrals or Radon measures. The last part of this paper includes some interesting examples.
A. Rosenthal (Lafayette, Ind.).

Viola, Tullio. Funzioni quasi continue in spazi astratti. *Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat.* (8) 18 (1955), 145-147.

The quasi-continuous functions of M. Picone [M. Picone and T. Viola, *Lezioni sulla teoria moderna dell'integrazione*, Einaudi, Turin, 1952, Chap. VI; *MR* 14, 256] are given an abstract setting by assuming that the space S of points P and the value space S' of the function $f(P)$ are topological spaces, while the value space S' of the mass or set function $\mu(e)$ is an additive metric complete space. The function μ is defined on the family \mathfrak{F} of Borel subsets of S , is completely additive and of bounded variation in terms of the metric on S' for all subsets of \mathfrak{F} . In order to obtain theorems similar to those for the real-valued case, the spaces S and S' are usually assumed to be metric and separable.
T. H. Hildebrandt.

See also: Cafiero, p. 719; Ul'yanov, p. 732; Tagamlicki, p. 767; Aquaro, p. 771; Talmanov, p. 772.

Theory of Measure and Integration

Cafiero, Federico. Sulla teoria della misura in un insieme astratto. *Ann. Mat. Pura Appl.* (4) 40 (1955), 269-283.

The author studies a definition of additivity, proposed by Caccioppoli, for a finite real-valued function of sets $H \in C$, where the class C , which consists of certain subsets of a set $S \in C$, is not necessarily additive, but is closed under finite union and countable intersection and satisfies the following condition: to each $H \in C$ there exist in C sequences $\{H_k\}, \{H'_k\}$ where $H_k \supset S - H'_k \supset H_{k+1}$ ($k=1, 2, \dots$) such that $H = \lim H_k$. (Incidentally the empty set $H_k' \setminus H_{k+1}$ then belongs to C .) He shows that a function additive in this sense coincides with the restriction to C of an additive function in the usual sense (defined in an additive class).
L. C. Young (Madison, Wis.).

de Finetti, Bruno. Sulla teoria astratta della misura e dell'integrazione. *Ann. Mat. Pura Appl.* (4) 40 (1955), 307-319.

The author proposes, as a proper setting for integration theory, the study of functionals μ satisfying the following conditions: let \mathfrak{F} denote the linear space of all real-valued bounded functions on a set \mathfrak{E} , let $\mathfrak{F}^{(+)}$ denote the subset of \mathfrak{F} consisting of functions with positive lower bound, and let \mathfrak{L} designate a linear subspace of \mathfrak{F} . An integral μ is then defined as a real-valued function on \mathfrak{L} which i) is additive, ii) is positive on $\mathfrak{F}^{(+)}$, iii) has the value 1 on the function identically equal to 1 on \mathfrak{E} .

Measure and integral are related in the usual way by the use of characteristic functions of sets. The usual more restrictive property of countable additivity is then to be regarded as a sort of continuity of a measure defined in the above sense on certain linear subspaces of \mathfrak{F} .

The extension of μ to all of \mathfrak{F} , this notion of continuity, compatibility and invariance of measures are discussed briefly. Geometrical interpretations figure prominently and in particular considerable use is made of the partition of \mathfrak{F} into the set $\mathfrak{F}^{(+)}$, the set $\mathfrak{F}^{(-)}$ of functions with nega-

tive upper bound, and the residual set $\mathfrak{F}^{(2)}$ (termed neutral functions). The present paper is in the nature of an introductory sketch without much detail, and the study is to be continued in a subsequent publication.

W. R. Transue (Gambier, Ohio).

Morse, Marston. Bimeasures and their integral extensions.

Ann. Mat. Pura Appl. (4) 39 (1955), 345-356.

This paper aims at motivating and expounding (without proofs) some of the recent work of Morse and Transue, most of which has yet to appear [J. Analyse Math. 4 (1954/55), 149-186; MR 17, 469; see also the paper reviewed below]. In brief one finds here a comparison of two concepts, namely: (i) that of a so-called "bimeasure" on a product $E = E' \times E''$ of two locally compact spaces, and (ii) that of a (Radon) measure on E . Denoting by $\mathcal{K} = \mathcal{K}(E)$ the vector subspace of R^E formed of continuous functions with compact supports, and writing \mathcal{K}' , \mathcal{K}'' for $\mathcal{K}(E')$, $\mathcal{K}(E'')$ respectively, a bimeasure Λ on E is defined to be a bilinear form on $\mathcal{K}' \times \mathcal{K}''$ which is separately continuous for the product topology. (The factors \mathcal{K}' and \mathcal{K}'' are each topologised à la Bourbaki.) It is pointed out that the vector space of bimeasures is not a Riesz space, with the result that a Riesz decomposition of a bimeasure is generally impossible. As a consequence, the Bourbaki theory of Radon measures cannot be taken over directly for bimeasures. Nevertheless, a possible somewhat analogous process is outlined, leading to the concept of a pair $(x', x'') \in R^{E'} \times R^{E''}$ being integrable for Λ .

Reviewer's note. The third paragraph on p. 348 appears rather misleading. Whilst there is no sense in which a measure can be said to be literally a bimeasure (one being a linear form on \mathcal{K} , the other a bilinear form on $\mathcal{K}' \times \mathcal{K}''$), yet there is an obvious sense in which every measure generates in a natural fashion a corresponding bimeasure, the correspondence being 1-1. Indeed a bimeasure, being a bilinear form on $\mathcal{K}' \times \mathcal{K}''$, is canonically associated with a linear form λ on $\mathcal{K}' \otimes \mathcal{K}''$ in such a way that $\Lambda(x', x'') = \lambda(x' \otimes x'')$ identically. On the other hand, $\mathcal{K}' \otimes \mathcal{K}''$ can be identified with a (dense) vector subspace of \mathcal{K} by pairing off $\sum_i x'_i \otimes x''_i$ with the function (element of \mathcal{K}): $E' \times E'' \rightarrow R$ defined by $(t', t'') \rightarrow \sum_i x'_i(t') \cdot x''_i(t'')$. From this point of view the distinction between measures and bimeasures is naturally to be expressed in terms of continuity relative to various topologies on $\mathcal{K}' \otimes \mathcal{K}''$. Cf. the analogous situation involving measures and distributions, where again the Riesz decomposition marks a dividing line.

R. E. Edwards (London).

Morse, Marston; and Transue, William. The representation of a C-bimeasure on a general rectangle.

Proc. Nat. Acad. Sci. U.S.A. 42 (1956), 89-95.

The notation and terminology differ from that used in the preceding review; for the sake of space we continue with the latter. The principal result is a representation theorem for bimeasures Λ on $E = E' \times E''$, each of E' and E'' being a real interval. Unlike the paper reviewed above, this one admits complex-valued functions and bimeasures. The representation takes the form

$$\Lambda(x', x'') = \int_{E'} x'(t') d_{t'} \int_{E''} x''(t'') d_{t''} h(t', t'');$$

the integral is a Riemann-Stieltjes one, and h is a point-function on E with finite Fréchet variation over each compact sub-rectangle of E , and is regular (i.e. left-left continuous in a sense which is almost self-explanatory).

h may be chosen so that it has arbitrarily pre-assigned "null lines" $t' = t'_0$, $t'' = t''_0$, and it is then uniquely determined by Λ . There is a converse assertion. Finally, the Fréchet variation of h over E is shown to be equal to

$$\|\Lambda\| = \sup \{ |\Lambda(x', x'')| / \|x'\| \cdot \|x''\| : x' \in \mathcal{K}', x'' \in \mathcal{K}'' \},$$

where $\|x'\| = \sup \{ |x'(t')| : t' \in E' \}$ and $\|x''\|$ is similarly defined. This $\|\Lambda\|$ may be $+\infty$, unless both E' and E'' are compact.

R. E. Edwards (London).

Ishii, Tadashi. On semi-reducible measures. Proc. Japan Acad. 31 (1955), 648-652.

The Baire sets \mathfrak{B} of a topological space X is the smallest σ -algebra of sets containing all sets $\{x: x \in X, f(x) = 0\}$, where f is a continuous real-valued function on X . A (finite and countably additive) measure μ on \mathfrak{B} is semi-reducible in the sense of Katětov [Fund. Math. 38 (1951), 73-84; MR 14, 27] if there is a closed subset P of X such that $\mu(G) > 0$ for all open sets $G \in \mathfrak{B}$ such that $G \cap P \neq \emptyset$ and $\mu(F) = 0$ for all closed sets F such that $F \cap P = \emptyset$. Sample theorems: (A) Let X be a normal space. Then every Baire measure μ is semi-reducible if and only if every Baire measure on every proper closed subspace is semi-reducible. (B) Theorem (A) holds for 2-valued Baire measures on any completely regular space. (C) Let X be completely regular and every 2-valued Baire measure semi-reducible. Then the same is true for all F_σ -subsets of X . (D) Let X be paracompact, and suppose that no discrete closed subspace of X admits a non-trivial 2-valued measure. Then every Baire measure on X is semi-reduced by a closed set P that has the Lindelöf property.

E. Hewitt (Princeton, N.J.).

Nikodým, Otton Martin. Sur l'extension d'une mesure (qui peut être non archimédienne), simplement additive sur une tribu de Boole simplement additive à une autre tribu, plus étendue. IV. Extension de mesure dans le cas général. C. R. Acad. Sci. Paris 242 (1956), 864-866.

In general, the extension method used in part III [same C. R. 241 (1955), 1695-1696; MR 17, 594] requires adjoining of new elements to the range of the measure. Procedures to this end are discussed in the present note.

H. M. Schaefer (St. Louis, Mo.).

Brodskii, M. L. On arithmetic sums of sets contained in a given set. Ukrain. Mat. Ž. 4 (1952), 195-203. (Russian)

The paper treats the arithmetical sum $A+B$ and the Cartesian product $A \times B$ of two sets (the former consists of the elements $x+y$ where $x \in A$, $y \in B$). The author first remarks that there exists a perfect set M of reals such that the inclusion $M \supset A+B$ implies that A and B cannot both contain more than one point, and that this remark is a corollary of the following theorem: Let f_k denote, for a finite or countable system of k , a continuous function of the first n_k variables x_i ($a \leq x_i \leq b$; $i=1, 2, \dots$), not identically zero on any segment parallel to a coordinate axis; then on any segment of reals interior to (a, b) , there is a perfect set M such that no equation $f_k=0$ can hold for values of the n_k variables all belonging to M . The author's further results fall into the following patterns: in the space E , given a set A with the property P , there is a perfect set B such that $A+B$ has the property P' ; in the space E , given a set M with the property Q , there is a perfect set B such that $M \supset A+B$ for some set A with the property Q' ; given a set M with the property R in the space $E_1 \times E_2$, there is a perfect set B in E_2 such that

$M \supset A \times B$ for some set A with the property R in E_1 . The cases treated are: E, E_1, E_2 Banach spaces, P, P' =first category, Q, Q' =second category, R =second category and a Borelian class; E real line, E_1, E_2 Euclidean spaces, $\varepsilon > 0$ sufficiently small, P, P' =possession of measure $m \geq 0$, $m' \leq (1+\varepsilon)m$, Q, Q' =possession of measure $m > 0$, $m' > m - \varepsilon$ and also Q, Q' =inclusion (mod. nullset) of interval of measure $m > 0$, $m' > m - \varepsilon$, R =possession of positive measure, and also R =inclusion (mod. nullset) of an interval.

L. C. Young.

Dubrovskii, V. M. On the best majorant of a family of completely additive set functions. *Moskov. Gos. Univ. Uč. Zap.* 163, Mat. 6 (1952), 89-98. (Russian)

V is a majorant of a family \mathfrak{F} of completely additive complex-valued set functions on a σ -algebra \mathfrak{M} if $|\phi(e)| \leq V(e)$ for every $\phi \in \mathfrak{F}$, $e \in \mathfrak{M}$; U is a best majorant if U is a majorant such that $U(e) \leq V(e)$ for every $e \in \mathfrak{M}$ and every majorant V . The author shows that the best majorant $U(e)$ always exists and has the (perhaps infinite) value

$$\sup \sum_{i=1}^n \sup_{\phi \in \mathfrak{F}} |\phi(e_i)| = \sup \sum_{i=1}^{\infty} \sup_{\phi \in \mathfrak{F}} |\phi(e_i)|,$$

where the sup before the finite (infinite) series is taken over all finite (countable) decompositions of $e \in \mathfrak{M}$ into disjoint unions of $e_i \in \mathfrak{M}$. In particular, if A and B are subsets of Euclidean n -space, of finite non-zero Lebesgue measure, if \mathfrak{M} is the class of all Lebesgue measurable subsets of B , and if \mathfrak{F} is the class of all $\phi(e; x) = \int_e f(x, y) dy$, $x \in A$, for some fixed Lebesgue measurable complex-valued f on $A \times B$, integrable in y for each $x \in A$, then $U(e) \geq \int_e \text{ess sup}_{x \in A} |f(x, y)| dy$; furthermore, there exists $A^* \subset A$ such that $\text{meas}(A - A^*) = 0$ and the best majorant $U^*(e)$ of $\{\phi(e; X)\}_{X \in \mathfrak{M}^*}$ has the value $\int_{A^*} \text{ess sup}_{x \in A^*} |f(x, y)| dy$. If, in addition,

$$\text{meas}\{y: f(x, y) \text{ is continuous on } A\} = \text{meas } B,$$

then $U(e) = \int_e \text{ess sup}_{x \in A} |f(x, y)| dy$.

A. E. Livingston (Seattle, Wash.).

Zaslavskii, I. D. On R -integrability of the superposition of R -integrable functions. *Vestnik Leningrad. Univ.* 1953, no. 11, 49-55. (Russian)

Integrability will refer always to Riemann integrability. The author is concerned, then, with the characterization of the following classes of functions. (a) The class R^* of functions $f(x)$ defined on $[a, b]$ which have the property that for any function $\varphi(t)$ defined and integrable on any interval $[p, q]$ and with $a \leq \varphi(t) \leq b$ there, the composite function $f[\varphi(t)]$ will be integrable on $[p, q]$. (b) The class R_* of functions $\varphi(t)$ defined on $[p, q]$ which have the property that for any function $f(x)$ defined and integrable on the interval $[a, b]$, where $a = \inf \varphi(t)$, $b = \sup \varphi(t)$, $p \leq t \leq q$, the composite function $f[\varphi(t)]$ will be integrable on $[p, q]$.

Results are as follows: (1) R^* consists of the continuous functions on $[a, b]$. (2) R_* consists of the integrable functions on $[p, q]$ such that for every closed set $F_x \subset [a, b]$ of measure zero, the set $V_\varphi \cdot \Pi_\varphi(F_x)$ is of measure zero, where V_φ denotes the set of points of $[p, q]$ which are not interior (relative to $[p, q]$) to an interval on which φ is constant, and $\Pi_\varphi(F_x)$ denotes the complete inverse image under φ of F_x . (3) If $\mu(x)$ denotes the measure of the product of V_φ and the set on which $\varphi(t) \leq x$, a second characterization of R_* is as those integrable functions on $[p, q]$ for which $\mu(x)$ is absolutely continuous on $[a, b]$ with $\mu(a) = 0$. (4) A function analytic on $[p, q]$ belongs to R_* .

The proofs for the most part use the fact that a function is Riemann integrable if and only if its set of points of discontinuity is of measure zero.

W. R. Transue.

McShane, E. J. On Stieltjes integration. *Proc. Amer. Math. Soc.* 7 (1956), 69-74.

This note extends to spaces of more than one dimension the theorems on Stieltjes integrals which assert that if the integral $\int f dg$ exists then f and g have no common discontinuities and the integral $\int f dt$ exists also, where t is the total variation function of g . In making the proof a modified Stieltjes integral defined by the author and Botts [Duke Math. J. 19 (1952), 293-302; MR 13, 924] is used to advantage.

T. H. Hildebrandt.

Stolyarov, N. A. On a generalization of the Stieltjes integral. *Dokl. Akad. Nauk SSSR (N.S.)* 105 (1955), 652-655. (Russian)

H. Hahn [Akad. Wiss. Wien. S.-B. IIa. 134 (1925), 449-470] and L. V. Kantorovič [C.R. (Dokl.) Acad. Sci. URSS (N.S.) 4 (1934), 417-421] have considered the following definition: let f and φ be real functions defined and finite on an interval $[a, b]$. If for $a = x_0 < x_1 < \dots < x_n = b$ the sum

$$S = \sum_{i=1}^{n-1} f(x_i) \{[\varphi(x_{i+1}) - \varphi(x_i)](x_{i+1} - x_i)^{-1} - [\varphi(x_i) - \varphi(x_{i-1})](x_i - x_{i-1})^{-1}\}$$

approaches a limit as $\max_i \{x_{i+1} - x_i\}$ tends to zero, this limit is called a generalized Stieltjes integral and designated by $\int_a^b f(x) d^2 \varphi(x)/dx$ [which reduces, in case φ has a continuous second derivative, to $\int_a^b f(x) \varphi''(x) dx$].

The present note observes that summation by parts, applied to the sums used in defining $\int_a^b f(x) d^2 \varphi(x)/dx$ and $\int_a^b \varphi(x) d^2 f(x)/dx$, relates each of these sums to that used in defining the Hellinger integral $\int_a^b df(x) d\varphi(x)/dx$. The author uses this fact to derive relations between these three integrals under suitable conditions on f and φ . An example is the following formula for integration by parts, valid if one of the integrals is known to exist and if the functions f and φ have right-hand derivatives $f_+(a)$, $\varphi_+(a)$ at a and left-hand derivatives $f_-(b)$, $\varphi_-(b)$ at b .

$$\int_a^b f(x) \frac{d^2 \varphi(x)}{dx} = f(b) \varphi_-(b) - f(a) \varphi_+(a) - f_-(b) \varphi(b) + f_+(a) \varphi(a) + \int_a^b \varphi(x) \frac{d^2 f(x)}{dx}.$$

[The author's notation for these derivatives interchanges the $+$ and $-$ signs.]

W. R. Transue (Gambier, Ohio).

Krickeberg, Klaus. Charakterisierung oberer und unterer Integrale durch Additivitäts- und Mittelwertigenschaften. *Math. Z.* 61 (1955), 374-385.

This paper, which is the author's "Habilitationsschrift" at Würzburg University, continues the investigation of upper and lower integrals, contained in his previous paper [Math. Nachr. 9 (1953), 86-128; MR 14, 735, 1278]. H. Hahn defined the integral of a real-valued measurable function as a totally additive set function with the fundamental mean-value property. Since the upper and lower integrals of any real-valued (not necessarily measurable) function have the same properties, the author asks whether a similar characterization of the upper and lower integrals is possible. Let μ be a measure defined in a σ -field \mathfrak{S} and let \mathfrak{M} be the system of the zero-

sets in \mathfrak{S} . Let \mathfrak{S} contain a maximal set E on which the real function f be defined. The following is easily seen: If the upper integral $\int_E f d\mu$ exists, then $\varphi_0(M) = \int_M f d\mu$ (with $M \in \mathfrak{S}$) is the greatest one among the totally additive set functions φ defined in \mathfrak{S} and satisfying the inequality $\varphi(M) \leq \int_M f d\mu$. Analogously for the lower integrals. The main problem discussed in this paper is the question under what additional conditions conversely the existence of $\int_E f d\mu$ follows. The author gives such a condition: For every "Borel ideal" \mathfrak{I} in \mathfrak{S} , which contains \mathfrak{R} , and for every set $K \in \mathfrak{S} - \mathfrak{I}$ there exists a finite measure λ defined in \mathfrak{S} , such that $\lambda(I) = 0$ for every $I \in \mathfrak{I}$ and $\lambda(K) > 0$. Here "Borel ideal" \mathfrak{I} in \mathfrak{S} means a σ -field $\mathfrak{I} \subseteq \mathfrak{S}$ such that with every set $I \in \mathfrak{I}$ also every subset of I , which is contained in \mathfrak{S} , belongs to \mathfrak{I} . The preceding condition is illustrated by some interesting examples.

A. Rosenthal (Lafayette, Ind.).

Łojasiewicz, S. Sur la formule de Green-Gauss-Ostrogradsky. Ann. Polon. Math. 1 (1955), 306-325.

The author proves a generalized form of the Green-Gauss theorem. Let G be an open set in R^n whose frontier (apart from a subset of $(n-1)$ -dimensional measure zero) is defined parametrically over a set Ω by $X = X(u)$, where $X \in R^n$, $u \in R^{n-1}$. Write $b_i(u)$ for the Jacobian $\partial(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) / \partial(u_1, \dots, u_{n-1})$, $a_i(u) = (-1)^{i+1} b_i(u)$, $A(u) = (a_1(u), \dots, a_n(u))$. For a vector function $F(x) = (f_1(x), \dots, f_n(x))$ defined over G and of class C' we have

$$\int_G \operatorname{div} F dx = \int_{\Omega} F(x(u)) \cdot A(u) du$$

provided certain conditions on G are satisfied. The method is to study the solutions of the differential equation $\dot{x} = F(x)$ in a domain containing G . The reviewer was not able to follow all the details of the argument.

H. G. Eggleston (Cambridge, England).

See also: Grošev, p. 717; Hayes and Pauc, p. 719; Sebastião e Silva, p. 766.

Functions of a Complex Variable, Generalizations

Parodi, Maurice. Sur deux propriétés des polynômes. C. R. Acad. Sci. Paris 242 (1956), 598-600.

The author presents the following two Pellet-type theorems which he derives with the aid of a result due to A. Brauer [Duke Math. J. 13 (1946), 387-395; MR 8, 192]. Let $b^2 = \sum_{k=1}^n |a_k| > 1$. Then, (I) if $|a_1| > 2b$, then $f(z) = z^n + a_1 z^{n-1} + \dots + a_n$, $a_n \neq 0$, has one and only zero in the circle $K: |z + a_1| \leq b$. (II) If $|a_1| > 1 + b^2$, then $f(z)$ has $n-1$ zeros inside the unit circle and its one zero outside the unit circle lies inside the circle $|z + a_1| \leq 1$.

M. Marden (Milwaukee, Wis.).

Lukomskaya, M. A. On division of power series. Belorussk. Gos. Univ. Uč. Zap. Ser. Fiz.-Mat. 15 (1953), 46-48. (Russian)

If $\sup |a_n|^{1/n} = M$ for $f(z) = 1 - \sum_{n=1}^{\infty} a_n z^n$, then $f(z)$ has no zeros in the circle $|z| < 1/2M$. This is proved by considering the power series for $1/f(z)$. A. W. Goodman.

Ricci, Giovanni. Prolungabilità e ultraconvergenza delle serie di potenze. Modulazione del margine delle lacune. II. Boll. Un. Mat. Ital. (3) 10 (1955), 439-452.

The author defines the relative separation (scarto relativo) between two increasing sequences $\{m_k\}$ and $\{n_k\}$ of integers. For $\delta > 1$ the symbol $\{n_k\}_\delta$ denotes the set of all integers n which satisfy the condition $n_k/\delta < u < n_{k+1}/\delta$ for at least one index k . The relative separation $\Lambda\{m_k, n_k\} = \Lambda\{n_k, m_k\}$ is defined as the supremum of the nonnegative quantities $\delta - 1$ for which the two sequences $\{m_k\}$ and $\{n_k\}_\delta$ have at most finitely many elements in common.

The construction of a certain class of overconvergent Taylor series leads to the following extension of a theorem of Bourion [Ann. Sci. Ecole Norm. Sup. (3) 50 (1933), 245-318, see p. 274]. If $\{n_k\}$ is an increasing sequence of integers, there exists a power series $f(z) = \sum a_n z^n$ such that the sequence of partial sums $s_{n_k}(z)$ overconverges (in the neighborhood of every regular point on the circle of convergence) for each sequence $\{m_k\}$ for which $\Lambda\{m_k, n_k\} > 0$, and for no other sequence $\{m_k\}$. G. Piranian.

Blambert, Maurice. Sur les points singuliers des séries de Dirichlet d'une classe de Cramer. Ann. Sci. Ecole Norm. Sup. (3) 72 (1955), 199-235.

Gegeben sei die Dirichletreihe $\varphi(s) = \sum b_p e^{-\mu_p s}$ ($s = \sigma + it$) mit der Konvergenzabszisse $\sigma_C < \infty$, und eine ganze Funktion $\theta(z) = \sum \gamma_r (\Gamma(r+1))^{-1} z^r$. Pólya [Nachr. Ges. Wiss. Göttingen. Math.-Phys. Kl. 1927, 187-195; vgl. auch Blambert, C.R. Acad. Sci. Paris 237 (1953), 1622-1624; MR 15, 518] behandelt das Problem, bei Kenntnis der Singularitäten von $\varphi(s)$ auf die Lage der Singularitäten von $\sum (-1)^r (\Gamma(r+1))^{-1} \gamma_r \varphi^{(r)}(s)$ zu schließen. Der Verf. setzt $\psi(s) = \sum b_p \theta(\mu_p) e^{-\mu_p s}$ und untersucht in 6 Sätzen die Singularitäten von $\psi(s)$ bei Kenntnis derjenigen von $\varphi(s)$. Als Beispiel erwähnen wir Theorem 1. Die Folge $\{\mu_p\}$ habe eine Dichte D ($0 < D < \infty$) und $\varphi(s)$ sei über die Regularitätsgerade $\sigma = \sigma_H$ hinaus fortsetzbar. $\theta(z)$ sei vom Exponentialtyp 0, und alle Nullstellen (außer endlich vielen) seien < 0 . Ist dann β mit $\Re(\beta) = \sigma_H$ singular für $\varphi(s)$, so auch für $\psi(s)$. D. Gaier (Stuttgart).

Gahov, F. D.; and Čibrikova, L. I. On Riemann's boundary problem for the case of intersecting contours. Kazan. Gos. Univ. Uč. Zap. 113, no. 10 (1953), 107-110. (Russian)

Let L be a contour in the z -plane consisting of a finite collection of closed and open piecewise smooth arcs having a finite number of common points and let $g(t)$, $G(t)$ be functions given on L satisfying a Hölder condition and $G(t) \neq 0$. Let it be required to find a piecewise holomorphic function $\Phi(z)$ which has right and left hand limits on L satisfying $\Phi^+(t) = G(t)\Phi^-(t) + g(t)$ except at points of intersection and at endpoints of L where $\Phi(z)$ may become infinite of order < 1 and of infinitely small order, respectively. The solution of this Riemann-Hilbert problem is obtained by quadratures from the "canonical solution" $X(z)$ which is a particular solution of $X^+(t) = G(t)X^-(t)$ vanishing at no ordinary point of L and of highest possible order at ∞ . The authors construct $X(z)$ as follows: Let t_k be an arbitrary point on the closed curve $L_k \subset L$ or an endpoint of the open L_k ; let

$$\log G(t_k - 0) - \log G(t_k + 0) = 2\pi i \chi_k;$$

$$\Gamma(z) = \frac{1}{2\pi i} \int_L \frac{\log G(\tau)}{\tau - z} d\tau.$$

Then $X(z) = \Pi(z - t_k)^{-\chi_k} e^{\Gamma(z)}$. This is a simpler expression for $X(z)$ than that found by Kveselava [Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze 17 (1949), 1-27; MR 13, 135]. M. Golomb.

Velez Cantarell, Francisco. Asymptotic developments in factorial series. Collect. Math. 7 (1954), 141-191. (Spanish)

The author develops a theory of asymptotic factorial series. The theory shows the expected analogies with the theory of asymptotic power series on the one hand and the theory of convergent factorial series on the other hand. Differentiation, integration, and the uniqueness problem are discussed. The author illustrates the theory by discussing in detail an everywhere divergent factorial series associated with a special convergent Laplace integral. R. P. Boas, Jr. (Evanston, Ill.).

*** Meili, Heino Jürg.** Über das Eindeutigkeitsproblem in der Theorie der asymptotischen Reihen. Dissertation, Universität Zürich, 1954. 46 pp.

Starting with Poincaré's idea of asymptotic series, the thesis is concerned with necessary and sufficient conditions under which the asymptotic expansion about a boundary point, of a function regular in a region, may be unique in the sense that it represents only one function. Let $f_1(w)$, $f_2(w)$ be two functions regular in a region G , w_0 a boundary point of G , and k_1, k_2 positive constants. What necessary and sufficient conditions must the a_n satisfy in order that the relations

$$(*) \quad |f_1(w) - \sum_{n=0}^{\infty} c_n(w-w_0)^n| / (w-w_0)^n \leq k_n a_n^* \quad (s=1, 2; n=0, 1, 2, \dots; w \in G)$$

imply that $f_1(w) = f_2(w)$? Writing $F(w)$ for $f_1(w) - f_2(w)$, Carleman exhibited (*) in the simpler form

$$(**) \quad |F(w) / (w-w_0)^n| \leq k_n a_n^* \quad (n=0, 1, 2, \dots),$$

and solved the problem of uniqueness for the circle $|w| < 1$ [Les fonctions quasi analytiques, Gauthier-Villars, Paris, 1926, Ch. V]. He found (i) a necessary and sufficient condition for uniqueness to be the divergence of $\sum_{n=0}^{\infty} 1/a_n^*$, where $a_n^* = \min_{n \geq r} a_r$ ($n=0, 1, 2, \dots$); (ii) if the series is convergent then a function $F(w)$, regular in the region, exists which is not identically zero and satisfies the conditions (**), and he then showed that the conditions for the uniqueness problem in the half-plane $\Re(w) > 0$ are the same as for the circle. Employing the essentially simpler methods which Ostrowski has used in the case of the circle [Acta Math. 53 (1929), 181-266], the author has given an elegant solution of the problem for the half-plane.

The main object of the thesis, however, is to extend these results from the half-plane to a more general region. Each of the regions considered is simply connected, bounded by a Jordan curve and has the origin as a boundary point, but while the first two regions considered are symmetric with regard to the real axis, the third is a Warschawski region [Math. Z. 35 (1932), 321-456]. The first region is a conformal representation of the half-plane in which the origins correspond, and the second region is an application of this derived from the strip region arising in the Ahlfors distortion theorems [Acta Soc. Sci. Fenn. Nova Ser. A. 1 (1930), no. 9]. The strip region is contained in an infinite strip of breadth π bisected by the real axis. The transformation $z = -\log w$ conformally represents the latter on the right half-plane with the representation of the strip region in its interior. In each of these cases the series in condition (i) is more general than in the case of the half-plane problem, while in the case of the Warschawski region the condition is the same as in (i). As with Ostrowski's work it is shown that the series in (i) can be replaced in all these cases by an integral. The way in which the

character of the region is reflected in the form of the condition (i) is a matter of some interest and, in a certain sense, the thesis may be regarded as a study of the interplay between these two. R. Wilson (Swansea).

Dvorkin, B. S. On a case of expansion of a function of a complex variable in a convergent Newton series. Stavropol. Gos. Ped. Inst. Sb. Nauč. Trud. 1952, no. 8, 145-153 (1953). (Russian)

The author studies the expansion of an entire function in a Newton interpolation series based on the points $\log_s(l+n)$, $n=1, 2, \dots$, where \log_s is the s th iterated logarithm and l is chosen so large that all the logarithms are positive. He shows that a sufficient condition for the expansion to converge is that $M(r) \leq \exp(h \log d) \exp_s(r/d)$ with $d > 3$, $0 < h < 1$, and \exp_s denoting the s th iterated exponential. R. P. Boas, Jr. (Evanston, Ill.).

Al'per, S. Ya. On some sequences of analytic functions forming complete systems and bases. Rostov. Gos. Univ. Uč. Zap. Fiz.-Mat. Fak. 32 (1955), no. 4, 9-13. (Russian)

The author uses a theorem of the reviewer [Trans. Amer. Math. Soc. 48 (1940), 467-487; MR 2, 80] on the expansion properties of a system of analytic functions that are "close" to z^n to prove that certain systems are complete and deduces the corresponding uniqueness theorems for entire functions of exponential type. Let $f(z) = \sum_{k=0}^{\infty} b_k z^k / k!$ be an entire function of exponential type with $b_k \neq 0$ and $\limsup |b_{k+1}/b_k| = \tau < \infty$ (generalization of e^{τ}). (1) Let $\{n_i\}$ be a strictly increasing sequence of positive integers, $|\zeta_i| \leq 1$, $\zeta_i \neq \zeta_j$ for $i \neq j$; then $\{z^{n_i} f^{(n_i)}(\zeta_i)\}$

is complete in $|z| \leq \tau^{-1} \log 2$. (2) If $|\zeta_k^{(n)}| \leq 1$ and $\zeta_i^{(n)} \neq \zeta_j^{(n)}$ for $i \neq j$, then the divided differences $\Delta_n[f(z); \zeta_0^{(n)}, \zeta_1^{(n)}, \dots, \zeta_k^{(n)}]$ form a basis in $|z| < \tau$, $\tau \leq \tau^{-1} \log 2$. (3) Same conclusion for

$$z^{n-k} \Delta_k[f^{(n)}(z); \zeta_0^{(n)}, \dots, \zeta_k^{(n)}].$$

R. P. Boas, Jr. (Evanston, Ill.).

Eremin, S. A. On a basis (in the wide sense) of the space of analytic functions. Kulbyšev. Indust. Inst. Sb. Nauč. Trudy. 1953, no. 4, 20-26. (Russian)

Consider the linear metric space E_R of functions analytic in $|z| < R$, as defined by Markušević [Mat. Sb. N. S. 17(59) (1945), 211-252; MR 7, 425]. The author asks when the sequence $\{z^n + f^{(n)}(z)\}$ forms a basis (in the wide sense) in E_R (i.e., a fundamental sequence with a bi-orthogonal sequence of linear functionals providing a unique formal expansion for every element). If $f(z) = \sum_{m=0}^{\infty} a_m z^m / m!$, the author's sufficient condition is that $\sum |a_n| b_n < 1/4$, where b_n are the partial sums of the series for e ($=e^1$). The proof depends on setting up explicitly a sequence of linear operators $S_{k,n}$ such that $S_{k,n}[f(z)] = z^k/k! + T_{k,n}$, where $\|T_{k,n}\| \rightarrow 0$ as $n \rightarrow \infty$.

R. P. Boas, Jr. (Evanston, Ill.).

Denjoy, Arnaud. L'allure asymptotique des fonctions entières d'ordre fini. C. R. Acad. Sci. Paris 242 (1956), 213-218.

Suppose that Γ is a path going to infinity in the open z -plane, that $f(z)$, $P(z)$ are integral functions and that

$$(*) \quad f(z) - P(z) \rightarrow 0, \text{ as } z \rightarrow \infty \text{ along } \Gamma.$$

Then the author calls $P(z)$ an asymptotic approximation

to $f(z)$ on Γ . He now proves the following theorem announced by him nearly 50 years ago [C.R. Acad. Sci. Paris 145 (1907), 106–108]: "An integral function of finite order ρ can have at most 2ρ distinct asymptotic approximations $P_j(z)$, of orders less than $\mu = [2 + 1/\rho]^{-1}$, on rectilinear paths Γ_j ." This theorem is an immediate consequence of the following lemma: "If $f(z)$ has asymptotic approximations $P_1(z)$, $P_2(z)$ of orders less than μ on two rays L_1 , L_2 which make an angle less than π/ρ with each other, then $P_1(z) = P_2(z)$."

The author also proves a stronger result, when the paths are congruent logarithmic spirals and shows further that (*) need only be satisfied in a sequence of annuli $r_n < |z| < r'_n$ when $r_n \rightarrow \infty$ and $(\log r'_n)/(\log r_n)$ is sufficiently large.

Ahlfors [Acta Soc. Sci. Fenn. Nova Ser. A. 1 (1930), no. 9] extended the author's lemma and hence the main theorem to the case of general paths instead of rays in the case when the $P_j(z)$ are constants, i.e. asymptotic values. The author now conjectures the possibility of such an extension in the present more general case, but this is far from obvious except when the $P_j(z)$ are polynomials. The question as to whether μ can be increased is also left open.

W. K. Hayman (Exeter).

Avetisyan, A. On a generalization of a theorem of G. Pólya. Dokl. Akad. Nauk SSSR (N.S.) 105 (1955), 885–888. (Russian)

The author generalizes Pólya's theory of the indicator diagram and the Borel transform to entire functions of finite order not less than $\frac{1}{2}$, provided that the indicator function $h(\theta)$ is nonnegative. He is evidently unaware that this has been done before; indeed, A. J. Macintyre [Proc. London Math. Soc. (2) 45 (1938), 1–20] did it in a way that does not require $h(\theta) \geq 0$.

R. P. Boas, Jr.

Lotockiĭ, A. V. Asymptotic value of Borel's function. Ivanov. Gos. Ped. Inst. Uč. Zap. Fiz.-Mat. Nauki 5 (1954), 71–72. (Russian)

The author considers an entire function $f(z) = \sum c_n z^n$ and its "associated Borel function" $\phi(z) = \sum c_n z^n/n!$. [It is more usual to call $z^{-1}f(1/z)$ the Borel transform of $\phi(z)$.] He proves that if $\phi(z) \rightarrow s$ (finite) along some ray, $f(z) \rightarrow s$ along the same ray. [This is effectively a special case of an Abelian theorem for Laplace transforms: see Doetsch, Theorie und Anwendung der Laplace-Transformation, Springer, Berlin, 1937, p. 188.]

R. P. Boas, Jr. (Evanston, Ill.).

Malliavin, Paul. Sur quelques procédés d'extrapolation. Acta Math. 93, 179–255 (1955).

This paper contains a wealth of results, most of which were announced previously [C.R. Acad. Sci. Paris 238 (1954), 2481–2483; 239 (1954), 20–22; MR 15, 942; 16, 18]. The central proposition is a uniqueness theorem for functions meromorphic in $x > 0$, with assigned zeros and poles there and satisfying an order condition on the lines $x = \text{const}$. The applications of this theorem give substantial improvements of known results in the whole field of problems covered by Mandelbrojt's "Séries adhérentes" [Gauthier-Villars, Paris, 1952; MR 14, 542]. A list of topics covered in the paper is given in the reviews cited above.

W. H. J. Fuchs (Ithaca, N.Y.).

Shah, Tao-Shing. On the moduli of some classes of analytic functions. Acta Math. Sinica 5 (1955), 439–454. (Chinese. English summary)

Let $f(z) = \sum_{n=1}^{\infty} a_n z^n$ be regular in the unit circle and let

$f(\zeta_1)/f(\zeta_2) \neq -1$ for any pair ζ_1, ζ_2 in the unit circle. Then $|f(re^{i\theta})| \leq r/(1-r^2)^{1/2}$ and this bound is sharp. The condition $f(\zeta_1)/f(\zeta_2) \neq 1$ leads to the same sharp bound on $|f(\zeta)|$. In this case Jenkins [Trans. Amer. Math. Soc. 76 (1954), 389–396; MR 16, 24] proved the same result by a different method. (Reviewed from the English summary.)

A. W. Goodman (Lexington, Ky.).

Goodman, A. W. Functions typically-real and meromorphic in the unit circle. Trans. Amer. Math. Soc. 81 (1956), 92–105.

The class TM of typically-real meromorphic functions $f(z)$ in $|z| < 1$ is defined by the characteristic property $\text{sign } \Im f(z) = \text{sign } \Im z$ and by the normalization $f(z) = z + \sum_{n=2}^{\infty} b_n z^n$ near $z=0$. It follows that such an $f(z)$ is real on the real axis, that all possible poles lie on it, are of order 1, and have negative residues. If z , with $|z| < 1$, is not real, then sharp upper bounds for $|f'(z)/\Im f(z)|$, $|f'(z)/f(z)|$, $|f(z)|$, and $|f'(z)|$ are given. More generally, for fixed non-real z , the exact set of all possible values $f(z)$ is a certain closed circle with centre on the imaginary axis and $z=0$ on the frontier. The extremal functions are always of the form $F_s(z) = z(1-2sz+z^2)^{-1}$ where s is real. The class TM can be reduced to the subclass TR of regular typically-real functions by a representation

$$f(z) = \mu t(z) + \sum m_j (p_j^{-2} - 1) \frac{z}{1 - (p_j + p_j^{-1})z + z^2},$$

where the sum is extended over all poles p_j , with residues $-m_j$, where $\mu = 1 - \sum m_j (p_j^{-2} - 1)$, and $t(z) \in \text{TR}$. If all $|p_j| \geq p > 0$, then the following sharp inequalities for the coefficients b_n hold: $|b_n| \leq B(n, p)$ when n is even, $\mu_n \leq b_n \leq B(n, p)$ when n is odd. Here

$$B(n, p) = (1 + p^2 + p^4 + \dots + p^{2n-2})/p^{n-1}$$

and $\mu(n) = \min(\sin n\theta/\sin \theta)$.

W. W. Rogosinski (Newcastle-upon-Tyne).

Tchakaloff, Lubomir. Sur une classe de fonctions analytiques univalentes. C. R. Acad. Sci. Paris 242 (1956), 437–439.

Let $f(z) = \sum_{k=1}^{\infty} A_k (z - a_k)^{-1}$, where $A_k \geq 0$, and where the a_k are distinct complex numbers. By an elementary argument it is shown that $f(z)$ is schlicht for $|z - z_0| > R\sqrt{2}$, where $|z - z_0| < R$ contains all the poles of $f(z)$. Hence $\int_{-\infty}^{\infty} \varphi(t) (z - e^{it})^{-1} dt$ is schlicht for $|z| > \sqrt{2}$, providing $\varphi(t) \geq 0$; the same result holds for $\int_{-1}^1 (z - t)^{-1} d\alpha(t)$ if $\alpha(t)$ is non-decreasing.

E. Reich (Santa Monica, Calif.).

Rogoŭin, V. S. Two sufficient conditions for univalence of a mapping. Rostov. Gos. Univ. Uč. Zap. Fiz.-Mat. Fak. 32 (1955), no. 4, 135–137. (Russian)

Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be convergent in $|z| > r$, and suppose that along every arc in $|z| > r$, $\Delta \arg f'(z) < \pi - 4 \arctan r$. Then $f(z)$ is univalent in $|z| > 1$.

Let D be a region with the property that every two points of D can be connected by a circular arc lying in D . Let $l(z_1, z_2)$ be the lower bound for the lengths of all such arcs joining z_1 and z_2 , and let $\varphi_1(z_1, z_2)$ be the lower bound of the central angles of such arcs. Let $\theta_0 = \sup \varphi_1(z_1, z_2)$ and $d_0 = \sup l(z_1, z_2)$ for z_1 and z_2 in D . If

$$\left| \frac{f''(z)}{f'(z)} \right| < \frac{\pi - \theta_0}{d_0}$$

in D , then $f(z)$ is univalent in D .

The author points out that $f(z) = \int_0^1 e^{-\alpha z} d\zeta$ is univalent

in $|z| < (\pi/2)^{1/2}$. This is somewhat larger than the value obtained by Nehari [Bull. Amer. Math. Soc. 55 (1949), 545-551; MR 10, 696].
A. W. Goodman.

Jenkins, James A. Sur quelques aspects globaux du théorème de Picard. Ann. Sci. Ecole Norm. Sup. (3) 72 (1955), 151-161.

Picard's theorem states that a function which is uniform and meromorphic in the neighbourhood of an essential singularity has at most two exceptional values. D. Dugué has considered the case of a function with n essential singularities and raised some questions concerning the relations among the values which are locally exceptional- P near the different essential points [same Ann. (3) 69 (1952), 65-81; MR 14, 741]. He proved that such a function can have at most $n+1$ distinct exceptional values at the n essential points. Such, in fact, is the case if the essential points are all isolated. Unfortunately, it is not true in general, and the error in the proof, which is not easy to locate, was discovered by J. A. Jenkins. Following consultations with Dugué he published the paper under review, and proved that there exists a function, meromorphic and uniform, with n essential points possessing two prescribed exceptional values at each essential point, giving in all $2n$ unrelated exceptional values. Further, he showed that there exists a function which may have at each of the n essential points either 0, 1 or 2 locally exceptional values, all of which are unrelated. In addition, certain deep questions raised by Dugué [loc. cit.] are dealt with. Two of these relate to the set of exceptional values of a uniform function which possesses an enumerable sequence of essential singularities with a single limit-point and to the set of exceptional values of a uniform function in the neighbourhood of a singular continuum. In the former case there exists a uniform function which has two locally exceptional- P values associated with each essential point and which are all distinct. In the latter case, for a function defined in a domain bounded by disjoint continua, the sets of exceptional values near the different parts of the boundary are independent. Another question arises as follows: it is known that if a meromorphic function $f(z)$ has only one value exceptional- P then there are two values of the constant c for which the zeros of $f(z)-c$ are all double zeros, and that if it has no values exceptional- P then there are four values of c for which this is so. An example of the former is $\cos z$ for which $c = \pm 1$, and of the latter is $\rho(z)$ for which $c = e_1, e_2, e_3$ or ∞ . These results are extended by the author to the case of a uniform and meromorphic function with n essential points and he shows that the values of the constants arising at the different points may be unrelated. The methods used throughout the paper involve a sophisticated treatment of the properties of Riemann surfaces and conformal representation.

R. Wilson (Swansea).

Dugué, Daniel. Note sur l'article précédent. Ann. Sci. Ecole Norm. Sup. (3) 72 (1955), 163.

The note opens with acknowledgements to J. A. Jenkins for his correction of the error in the paper cited in the preceding review. This is followed by a detailed explanation of the character of the error which, although revealing the tricky nature of the situation, is too complicated to be reproduced here. Finally, it should be noted that the particular functions used in the paper cited in the preceding review have an intrinsic value independent of the proof for which they were constructed. R. Wilson (Swansea).

Tsuji, Masatsugu. Remark on my former paper "On an extension of Löwner's theorem". Comment. Math. Univ. St. Paul. 4 (1955), 109-110.

L'A. reprend un article [Proc. Imp. Acad. Tokyo 18 (1942), 220-221; MR 7, 288] pour montrer que sa démonstration donne davantage. Il s'agit de $f(z)$ holomorphe dans le cercle-unité où $|f| < 1$. Tout sous ensemble borélien e de l'ensemble e_0 des points-frontière où f a une limite radiale de module 1, a, par cette fonction-limite, une image sur $|z| = 1$, qui est un ensemble analytique de mesure $-d\theta$ au moins égale à celle de e (et même plus grande si $0 < \text{mes } e_0 < 2\pi$ et $\text{mes } e > 0$).
M. Brelot (Paris).

Lokki, Olli. Über analytische Funktionen mit gegebenen Randwerten. Ann. Acad. Sci. Fenn. Ser. A. I. no. 202 (1955), 10 pp.

Let $\alpha(\varphi)$ denote an integrable function on $[0, 2\pi]$ satisfying $0 \leq \alpha(\varphi) < 1$. The problem of determining conditions on α which guarantee that there exists a function Φ analytic in $|z| < 1$ satisfying $\lim_{r \rightarrow 1} |\Phi(re^{i\varphi})| = \alpha(\varphi)$ a.e. is treated. The following condition is shown to be sufficient for the existence of a non-vanishing admitted Φ : $\int_0^{2\pi} \log |\log \alpha(\varphi)| d\varphi$ exists for some non-negative integer ν (\log , denotes the ν th iterated logarithm).

M. Heins (Providence, R.I.).

Safronova, G. P. Application of Orlicz metrics to some boundary problems of the theory of analytic functions. Dokl. Akad. Nauk SSSR (N.S.) 105 (1955), 222-224. (Russian)

Let $M(u)$, $N(u)$ be non-negative ($\neq 0$) convex for $u \geq 0$, each vanishing at $u=0$. A function f analytic in $|z| < 1$ is said to belong to the class H_M provided that

$$\int_0^{2\pi} M(|f(re^{i\varphi})|) d\theta = O(1).$$

A function $g(\theta)$ is said to belong to $L_N[0, 2\pi]$ provided that $\int_0^{2\pi} N(|g(\theta)|) d\theta < +\infty$. The following result generalizing one of Smirnov is established: If $f \in H_M$ and $f(e^{i\theta}) \in L_N[0, 2\pi]$, then $f \in H_N$. Related results are also given.
M. Heins (Providence, R.I.).

Tsuji, Masatsugu. A metrical theorem on conformal mapping. Comment. Math. Univ. St. Paul. 4 (1955), 111-112.

The following theorem is established. Let D be a bounded region, Γ its boundary, E (CT) a set of logarithmic capacity zero, each point of which is accessible. If the universal covering surface of D is mapped conformally onto $|z| < 1$, E is carried into an F_∞ of zero measure.
M. Heins (Providence, R.I.).

Pfugger, A. Über die Riemannsche Periodenrelation auf transzendenten hyperelliptischen Flächen. Comment. Math. Helv. 30 (1956), 98-106.

This paper considers transcendental hyperelliptic Riemann surfaces having the property that all the ramification points are real and positive. It is shown that a retrosection system dependent upon the metric structure of the surface may be introduced in such a manner that the Riemann bilinear formula for the inner product of quadratically integrable harmonic differentials appears as the assertion that the inner product in question is representable as the limit of a suitably chosen subsequence of partial sums of an infinite series whose terms are given in terms of integrals of the differentials along the retrosections of the system. M. Heins (Providence, R.I.).

Myrberg, Lauri. Über Abelsche Integrale mit unendlich vielen Singularitäten auf offenen Riemannschen Flächen.

Ann. Acad. Sci. Fenn. Ser. A. I. no. 209 (1955), 21 pp.

The author defines for a given non-compact Riemann surface F a class E of abelian differentials with the aid of the Bergman kernel vector. The class E enjoys the following properties: 1) E is a linear space. 2) Every abelian differential with at most a finite number of singularities which is quadratically integrable over each subsurface to which the singularities do not adhere belongs to E (the abelian differentials of Nevanlinna and Virtanen are subsumed in the class E). 3) Every differential of E which has a finite number of singularities is quadratically integrable over each subsurface to which the singularities do not adhere. 4) There exist differentials of the class E with an infinite set of singularities.

M. Heins (Providence, R.I.).

Myrberg, Lauri. Über das Dirichletsche Problem auf offenen Riemannschen Flächen. Ann. Acad. Sci. Fenn. Ser. A. I. no. 197 (1955), 11 pp.

Let F denote a non-compact Riemann surface and A a subsurface which is not relatively compact and whose relative boundary α consists of a finite or countably infinite set of analytic curves clustering at no point of F . Let f be a continuous real-valued function with domain α . It is shown that there exists a function continuous in \bar{A} , harmonic in A , whose restriction to α is f . The proof is achieved by first constructing a harmonic function u_1 in A which satisfies the desired boundary condition save at a countable set of points of α clustering at no point of F and having singularities of a simple type. Thereupon a compensating harmonic function u_2 is constructed in A which vanishes on α save at the points in question and is such that $u_1 - u_2$ satisfies the desired conditions.

M. Heins (Providence, R.I.).

Heins, Maurice. Lindelöfian maps. Ann. of Math. (2) 62 (1955), 418-446.

Le présent mémoire fait suite à Ann. of Math. (2) 61 (1955), 440-473 [MR 16, 1011] noté LP. Soit f une application d'une surface de Riemann F sur G , F ayant une frontière idéale positive. Le cas où G a une frontière idéale positive a été étudié dans LP, où l'on avait

$$S(p, q) = \sum_{f(r)=q} n(r) G_F(p, r) < +\infty \text{ (pour } f(p) \neq q),$$

$n(r)$ représentant la multiplicité de f en r et G_F la fonction de Green de F .

Sont étudiées ici les applications f telles que $S(p, q) < +\infty$ (pour $f(p) \neq q$) lorsque G a une frontière idéale nulle (applications Lindelöfiennes notées appl. L.). Une fonction caractéristique $T(\omega)$ ($\omega \in F$) du type de Nevanlinna peut alors être définie et les appl. L. sont précisément les applications de caractéristique bornée. De nombreux résultats sur les fonctions méromorphes de caractéristique bornée sont encore valables pour les appl. L. avec un minimum de restrictions sur F et G .

La fonction caractéristique $T(\omega)$ qui permet une extension du premier théorème fondamental de Nevanlinna au cas d'une application d'une surface de Riemann arbitraire non compacte sur G arbitraire, est obtenue à partir d'une fonction $D(s, q_1, q_2)$ harmonique en s sur G ayant deux singularités logarithmiques (+ et -) en q_1 et q_2 , et bornée en dehors d'un voisinage compact de q_1 et q_2 . L'auteur obtient aussi une fonction $s \rightarrow K(s, q)$ harmonique sur G ayant une singularité logarithmique en q et

bornée en dehors d'un voisinage de q , analogue à la fonction de Green lorsque G est à frontière idéale nulle.

Une très importante extension des résultats obtenus dans LP est ici atteinte par la fonction $q \rightarrow S^*(p, q)$ fonction limite supérieure de $q \rightarrow S(p, q)$. Alors la fonction $p \rightarrow S^*(p, q) - S(p, q)$ joue dans le cas actuel un rôle analogue à $w_q(p) = S^*(p, q) - S(p, q)$ de LP. L'harmonicité de $q \rightarrow S^*(p, q)$ en q_0 ($q_0 \neq f(p)$) caractérise les appl. L. qui sont du type Bl (cf. LP).

En désignant par O_L la classe des surfaces de Riemann n'admettant aucune fonction méromorphe L l'auteur obtient $O_{BA} \supset O_L \supset O_{BH}$, où $O_{BA} \neq O_L \neq O_{BH}$, et de plus montre l'existence d'une surface de Riemann pour laquelle il existe des fonctions méromorphes L mais pas de fonction holomorphe L . Cet important mémoire se termine par une intéressante application aux fonctions harmoniques différences de deux fonctions harmoniques non négatives.

L. Fourès (Marseille).

Wittich, H. Über eine Klasse Riemannscher Flächen.

Comment. Math. Helv. 30 (1956), 116-123.

The object of this paper is to study for certain classes of concretely given parabolic Riemann surfaces the Nevanlinna theory indices of the associated mapping functions. The surfaces are defined with the aid of the elliptic modular function and a simply-connected region of the upper half-plane which is symmetric with respect to the imaginary axis and whose boundary consists of an infinite set of so-called modular circles. The surfaces obtained are ramified only over 0, 1, ∞ , and each such surface possesses precisely one logarithmic ramification point (it lies over ∞). The associated mapping functions have lower order $\leq \frac{1}{2}$. There exist admitted surfaces of the following types: (1) the associated mapping function has arbitrarily large order; (2) the defect $\delta(\infty) = 0$; (3) the algebraic ramification index attains the maximum value one.

M. Heins (Providence, R.I.).

Stöhr, Alfred. Zur Funktionentheorie im Raum der symmetrischen Matrizen. Math. Z. 63 (1956), 464-477.

Let $W = f(Z)$ be a homogeneous linear mapping from the space of $n \times n$ symmetric matrices Z into the space of $m \times m$ symmetric matrices W ; the mapping is called rank-diminishing if $\text{rank } W \leq \text{rank } Z$ for all Z , rank-preserving if $\text{rank } W = \text{rank } Z$ for all Z , and special if it is of the form $W = A'ZA$ for a constant matrix A . The author shows that every special mapping is rank-diminishing, and is rank-preserving if and only if $\text{rank } A = n$; moreover, every rank-diminishing mapping which preserves the rank of at least one matrix is rank-preserving and special. If the mapping is rank-diminishing and if $E - W\bar{W} > 0$ whenever $E - Z\bar{Z} > 0$, then letting the eigenvalues of $Z\bar{Z}$ be $p_1^2 \geq \dots \geq p_n^2$ and those of $W\bar{W}$ be $q_1^2 \geq \dots \geq q_m^2$, $q_k^2 \leq p_k^2$ for $1 \leq k \leq \min(m, n)$ and $q_k^2 = 0$ for $n < k \leq m$ (if $n < m$).

A general differentiable mapping $W = \varphi(Z)$ (that is, a mapping which possesses a total derivative) defines a homogeneous linear mapping of the differentials dZ into the differentials dW ; φ is called differential-rank-diminishing or differential-special according as the associated linear mapping on the differentials is rank-diminishing or special. Examples are given of differential-special mappings, among which are the symplectic mappings of Siegel [Amer. J. Math. 65 (1943), 1-86; MR 4, 242]. The main result is the following analogue of the generalized Schwarz lemma: if $W = \varphi(Z)$ is an analytic

differential-rank-diminishing matrix function for which $E - W\bar{W} > 0$ when $E - Z\bar{Z} > 0$, ds is the symplectic metric in Z and $d\bar{z}$ that in \bar{W} , then $dt \leq ds$; moreover, the symplectic distance between $\varphi(Z_1)$ and $\varphi(Z_2)$ is at most equal to the symplectic distance between Z_1 and Z_2 . R. C. Gunning.

Lee, C. Y. Similarity principle with boundary conditions for pseudo-analytic functions. Duke Math. J. 23 (1956), 157-163.

The similarity principle of Bers for pseudo-analytic functions states essentially that to every pseudo-analytic function w on a domain D there corresponds an analytic function f such that w/f is bounded, bounded away from zero, and uniformly continuous on D . In the present paper it is shown that when D is bounded by a finite number of smooth curves, the ratio w/f can be made to satisfy certain boundary conditions. Applications are given to the construction of Green's function and L -measure (analogous to harmonic measure) for the elliptic partial differential operator

$$L(\varphi) = \varphi_{xx} + \varphi_{yy} + \alpha\varphi_x + \beta\varphi_y.$$

(A misprint of sign occurs in the definition of a_1 in Lemma 3.) M. G. Arsove (Seattle, Wash.).

See also: Denjoy, p. 717; Al'per, p. 729; Künzi, p. 734; Golomb, p. 742; Iacovache, p. 802.

Harmonic Functions, Potential Theory

Durand, Emile. Potentiel d'un disque uniformément chargé. C. R. Acad. Sci. Paris 242 (1956), 887-889.

The author establishes a formula for the potential of a disc at any point in three-space. A. E. Heins.

Topolyanskii, D. B. On a class of harmonic functions. Dnepropetrov. Gos. Univ. Nauč. Zap. 41 (1953), 165-168. (Russian)

A formulation in terms of line integrals of some obvious consequences of the bilinearity of the joint Dirichlet integral. M. G. Arsove (Seattle, Washington).

Rosati, Francesco. Proprietà integrali delle funzioni iperarmoniche in un dominio ellittico. Ricerche Mat. 4 (1955), 114-125.

The author introduces elliptic coordinates by the formulae

$$x = \left(e + \frac{1}{e}\right) \cos \vartheta, \quad y = \left(e - \frac{1}{e}\right) \sin \vartheta$$

and considers the hyperharmonic function expanded in Fourier series $\sum c_k(\varrho)e^{ik\vartheta}$. He treats first the biharmonic case. Putting $\Delta u = \bar{u}$, so that \bar{u} is a harmonic function, and using well-known properties of the Fourier coefficients of harmonic functions, an ordinary differential equation is deduced for $c_k(\varrho)$, using of course the expression for the Laplace operator Δu in the elliptic coordinates. The chief point is the study of the solutions of this equation. The method is then extended to the triharmonic and the n -superharmonic case. H. Bremekamp.

Bolotin, A. S. The inverse boundary problem for biharmonic functions. Kišinev. Gos. Univ. Uč. Zap. 11 (1954), 3-6. (Russian)

The author considers the problem of finding a domain

bounded by a curve C and satisfying the following conditions. A biharmonic function defined in the domain has prescribed boundary values together with its Laplacian, and the normal derivative of the Laplacian also takes on prescribed values. The problem is reduced to a similar problem for analytic functions discussed by Nužin [Kazan. Gos. Univ. Uč. Zap. 109 (1949)]. L. Bers.

See also: Lee, p. 727; Bers, p. 743; Fenchel, p. 778; Yeh, Martinek and Ludford, p. 796; Ashour, p. 808.

Series, Summability

Birindelli, Carlo. Qualche osservazione su alcuni generali procedimenti di sommazione delle serie. Ann. Mat. Pura Appl. (4) 39 (1955), 127-141.

In an earlier paper [same Ann. (4) 2 (1925), 263-295] the author introduced the following summation procedure. Let A be an unbounded subset of the space S_r and let there be defined on the set A a sequence $\{f_n(P)\}$ of real valued non-negative functions satisfying the conditions (1) $f_0(P) \geq f_1(P) \geq \dots \geq f_k(P) \geq \dots$, and (2) $\lim_P f_k(P) = 1$ ($k=0, 1, 2, \dots$). Set $F_n(P) = \sum_{k=0}^n f_k(P)u_k$. Then $g = \liminf_P \liminf_n F_n(P)$ and $\bar{g} = \limsup_P \limsup_n F_n(P)$ are defined respectively as the minimum and maximum sum of $\sum_{n=0}^{\infty} u_n$ relative to the sequence $\{f_n(P)\}$. The series $\sum_{n=0}^{\infty} u_n$ is spoken of as being either summable or non-summable on A according as either $g = \bar{g}$ or $g \neq \bar{g}$. Regarding this method of summation the author proved that the interval of indetermination (g, \bar{g}) is contained in the interval of indetermination of the ordinary summation process of convergence. In the present paper the author presents an extension of this procedure. Denote by A_P the subset of A consisting of those points Q of A which are not less distant from the origin than P . Set

$$F_n(P, Q) = \sum_{k=0}^n u_k f_k(P) f_k(Q), \quad E(P, Q) = \liminf_n F_n(P, Q) \\ \text{and} \quad \bar{F}(P, Q) = \limsup_n F_n(P, Q).$$

Also set $\psi(P) = \liminf_Q E(P, Q)$ (Q in A_P) and $\Phi(P) = \limsup_Q \bar{F}(P, Q)$ (Q in A_P). A summation procedure is then introduced as before by considering $\liminf_P \psi(P)$ and $\limsup_P \Phi(P)$. It is shown that the interval of indetermination of the extended method is contained in that of the original process. Extensive applications are given to the analytic continuation of the geometric series. V. F. Cowling (Lexington, Ky.).

Goffman, Casper; and Petersen, G. M. Submethods of regular matrix summability methods. Canad. J. Math. 8 (1956), 40-46.

A "submethod" of a regular (Toeplitz) matrix "method" A is a method of summability whose matrix is obtained by deleting a set of rows from A . A one-to-one correspondence is established between the submethods of A and the points of the interval $0 < \xi \leq 1$ by associating with each point ξ in this interval the submatrix of A whose n th row is deleted if, and only if, $a_n = 0$ in the non-terminating binary expansion $.a_1 a_2 \dots a_n \dots$ of ξ . The submatrix of A corresponding to ξ is denoted by $A(\xi)$, and a set of submethods $A(\xi)$, where $\xi \in E$, has a specific property whenever the set E has this property.

The following results are established. I. If A is a regular method, and $\{s_n\}$ is a bounded sequence which is not summable by A , then the set of submethods of A by which $\{s_n\}$ is summable is of the first category. II. For

every regular method A , the set of ξ for which $A(\xi)$ is equivalent (mutually consistent) to A is of the first category. III. For every regular method A there exist methods B and C , equivalent to A , such that B is equivalent to $B(\xi)$ for almost all values of ξ , and $C(\xi)$ is strictly stronger than C for almost all values of ξ . IV. The $(C, 1)$ -summability method is equivalent to almost all of its submethods. V. If A is a regular method, and $\{s_n\}$ is a bounded sequence not summable by A , then the set of submethods of A which sums $\{s_n\}$ is of measure either 0 or 1, and either value can occur. *R. G. Cooke.*

Kanno, Kōsi. On the Riemann summability. *Tōhoku Math. J.* (2) 6 (1954), 155–161.

Let s_n^p be the (C, β) -sum of the series $\sum_{n=1}^{\infty} a_n$. The author shows that if $s_n^p = o(n^\gamma)$ for $\beta > \gamma > 0$, $\gamma + 1 > p$, where p is a positive integer and $\sum_{n=1}^{\infty} |a_n|/n^\gamma = O(n^{\delta-1})$ for $\delta = p(\beta - \gamma)/(\beta + 1 - p)$ ($0 < \delta < 1$), then the series $\sum_{n=1}^{\infty} a_n$ is summable (R, p) to zero; i.e.,

$$\lim_{t \rightarrow 0} \sum_{n=1}^{\infty} a_n (\sin nt / nt)^p = 0.$$

For $p=1$, $\beta=1$, this reduces to a theorem of G. Sunouchi [same *J.* (2) 5 (1953), 34–42; MR 15, 304], and for $p=1$ to the theorem of H. Hirokawa and G. Sunouchi [ibid. 5 (1954), 261–267; MR 16, 240]. *G. Klein.*

Jakubík, Ján. Remark on absolutely convergent series. *Mat.-Fyz. Časopis. Slovensk. Akad. Vied* 5 (1955), 133–136. (Slovak. Russian summary)

Verfasser betrachtet die Menge W der Elemente eines B -Raumes X , die sich in der Gestalt $\sum c_i a_i$ darstellen lassen, wobei die a_i ($i=1, 2, \dots$) feste Elemente aus X mit $\|a_i\| < \infty$ sind und die c_i gewissen kompakten Mengen C_i komplexer Zahlen entnommen werden. Er fordert noch die Beschränktheit von $\bigcup C_i$ sowie, daß unendlich viele der C_i mehr als ein Element enthalten. Dann ist W perfekt. Dies verallgemeinert Resultate von P. K. Menon [Bull. Amer. Math. Soc. 54 (1948), 706–711; MR 10, 184] und von T. Šalát [Mat. Fyz. Časopis. Slovensk. Akad. Vied 4 (1954), 203–211; MR 16, 1099].

K. Zeller (Tübingen).

Šalát, Tibor. Remarks on Riemann's theorem on divergent series. *Mat.-Fyz. Časopis. Slovensk. Akad. Vied* 5 (1955), 94–100. (Slovak. Russian summary)

Sei $a_n > 0$, $a_n \rightarrow 0$ und $\sum a_n = +\infty$. Es bedeute X die Menge der Reihen $\sum \pm a_n$. Der Abstand zweier solcher Reihen sei $1/k$, wenn k der erste Index ist, bei dem die Reihen verschiedene Vorzeichenfaktoren besitzen [vgl. T. Šalát, derselbe *Časopis* 4 (1954), 203–211; MR 16, 1099]. Dann ist die Menge X_1 der Reihen aus X , die konvergieren oder bestimmt divergieren, von erster Kategorie und dicht in X . Die Menge der Reihen, die gegen eine bestimmte Summe m konvergieren, hat die Mächtigkeit des Kontinuums. *K. Zeller* (Tübingen).

Boyd, A. V. A Tauberian theorem for α -convergence of Cesàro means. *Proc. Amer. Math. Soc.* 7 (1956), 59–61.

In connection with the α -analogues of Cesàro and Abel summability introduced by Gehring [Trans. Amer. Math. Soc. 76 (1954), 420–443; MR 16, 346], the author obtains, in terms of the series $\sum na_n$, a necessary and sufficient condition for $(C, k; \alpha)$ -summability of a series $\sum a_n$ summable (A, α) , where $K \geq -1$, $0 \leq \alpha \leq 1$. *L. C. Young.*

Hsu, L. C. On an asymptotic integral. *Proc. Edinburgh Math. Soc.* (2) 10 (1956), 141–144.

An asymptotic representation of $\int_a^b \{f_n(x)\}^n g(x) dx$ is given, for large n , under several assumptions about the functions in the integrand. *T. E. Hull.*

See also: Lukomskaya, p. 722; Haplanov, p. 767.

Interpolation, Approximation, Orthogonal Functions

Gilenko, N. D. Representation of functions by series of polynomials with special coefficients. *Moskov. Gos. Ped. Inst. Uč. Zap.* 71 (1953), 63–69. (Russian)

In this paper the following problem is discussed: What should be the set of real numbers to which the coefficients of polynomials belong in order that a function, continuous in a given interval, could be expanded in an uniformly convergent series in those polynomials? The author proves two theorems:

(1) A function, continuous in an interval $(a, b) \subset (0, 1)$, can be expanded in an uniformly convergent series of polynomials, the coefficients of which belong to an arbitrary sequence of numbers, $\dots < c_{-2} < c_{-1} < 0 < c_1 < c_2 \dots$, where $\lim |c_n| = \infty$, as $n \rightarrow \infty$, and $\sup \{c_{n+1} - c_n\} \leq c$, $n=0, \pm 1, \pm 2, \dots$, and c is a constant.

(2) The set cannot be limited if the terms of the polynomials of degree higher than x^p , where p is a given natural number, are repeated in the series less than a certain number of times m . *S. Kulik* (Columbia, S.C.).

Tonyan, V. A. On weighted polynomial approximation of differentiable functions on the real axis. *Dokl. Akad. Nauk SSSR (N.S.)* 105 (1955), 656–658. (Russian)

Let $E_n(f, h)$ denote the best weighted polynomial approximation to $f(x)$ on $(-\infty, \infty)$ under the norm

$$\sup h(x) |f(x) - P_n(x)|.$$

The author obtains, by a method of Mergelyan [same *Dokl. (N.S.)* 97 (1954), 597–600; 101 (1955), 196; MR 16, 1104], an upper bound for $E_n(f, h)$ when $|f(x)| \leq 1$,

$\max_{(-x, x)} |f'(t)| \leq M(x)$, and $h(x) \leq \exp(-|x|^\lambda)$, $\lambda > 1$.

He states a similar result when (more generally) $x^{-1} \log\{1/h(x)\} \rightarrow \infty$ as $x \rightarrow \infty$.

R. P. Boas, Jr. (Evanston, Ill.).

Kipriyanov, I. A. On polynomials like S. N. Bernstein's for functions of two variables. *Kazan. Gos. Univ. Uč. Zap.* 113, no. 10 (1953), 193–207. (Russian)

Discussion of two-dimensional generalized Bernstein polynomials. The main result: let

$$\bar{B}_{n,m}(x, y) = \sum_{k=0}^n \sum_{l=0}^m f_{kl} p_{nk}(x) p_{ml}(y),$$

where $p_{nk}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, and f_{kl} is the integral mean

of $f(x, y)$ over the rectangle $k/(n+1) \leq x \leq (k+1)/(n+1)$, $l/(m+1) \leq y \leq (l+1)/(m+1)$, then (*) $\bar{B}_{n,m}(x, y) \rightarrow f(x, y)$ almost everywhere provided $n, m \rightarrow \infty$ and the ratio n/m does not approach 0 or ∞ . Since (*) is equivalent to $\partial^2 \bar{B}_{n,m}(F; x, y) / \partial x \partial y \rightarrow \partial^2 F / \partial x \partial y$, where

$$F(x, y) = \int_0^x \int_0^y f(u, v) du dv,$$

there is some connection with results of Butzer [Canad. J. Math. 5 (1953), 107-113; MR 14, 641] about partial derivatives of ordinary Bernstein polynomials $B_{n,m}(f;x,y)$. Some further types of generalized Bernstein polynomials are considered.
G. G. Lorentz (Detroit, Mich.).

Al'per, S. Ya. On uniform approximations of functions of a complex variable in a closed region. *Izv. Akad. Nauk SSSR. Ser. Mat.* 19 (1955), 423-444. (Russian)

Let D be a simply connected region bounded by a smooth Jordan curve $\Gamma: z=z(s)$, $0 \leq s \leq l$, s arc-length of Γ . Let $\theta(s)$ be the angle between the positive x -axis and the tangent to Γ at $z(s)$. Suppose that the modulus of continuity $\eta(t)$ of $\theta(s)$ satisfies $\int_0^t \eta(t) \log t |t^{-1} dt| < \infty$. The author investigates the best approximation in the Chebyshev sense in such regions D . He proves that for a function $f(z)$ analytic in D whose p th derivative satisfies a Lipschitz condition of order α ($0 < \alpha \leq 1$) in \bar{D} the best approximation in \bar{D} by polynomials of degree n is $O(n^{-p-\alpha})$. The approximation of $f(z)$ by partial sums of its expansion in Faber polynomials is studied, not only for ordinary convergence but also for certain methods of summability, including $(C, 1)$, the Bernstein-Rogosinski method and de la Vallée-Poussin's method. Typical results are: If $f(z)$ is continuous in \bar{D} with modulus of continuity $\omega(\delta)$, then the approximation by the n th partial sum is $O(\omega(1/n) \log n)$ and by the n th Bernstein-Rogosinski sum $O(\omega(1/n))$.
W. H. J. Fuchs.

Šaginyan, A. L. On the rapidity of polynomial approximation on arbitrary sets. *Akad. Nauk Armyan. SSR. Izv. Fiz.-Mat. Estest. Tehn. Nauk* 8 (1955), no. 3, 1-31. (Russian. Armenian summary)

Let E be a bounded closed set with simply connected complement. Let $w=\phi(z)$ be the function which maps CE conformally on $|w| > 1$, $\phi(\infty)=\infty$, $\phi'(\infty) > 0$.

The first part of the paper gives estimates for $|\phi(z)|$ in terms of geometrical quantities connected with the configuration of E . A typical result is as follows: Let E be in $|z| \leq d$, let $L: z=z(s)$ (s arc-length of L) be a closed, rectifiable curve in CE surrounding E and lying in $|z| \leq d_0$. Then for $z_0 \in L$

$$\log |\phi(z_0)| \geq (d_0/2d) \exp(-\int_L ds/\rho(s)),$$

where $\rho(s)$ is the distance of $z(s)$ from E . The method of proof is closely related to a method used by R. Nevanlinna [Eindeutige analytische Funktionen, 2nd ed., Springer, Berlin, 1953, Th. 5, p. 84; MR 15, 208]. In the second part of the paper the results obtained are applied to the uniform approximation by polynomials of functions analytic on E .

If $f(z)$ is analytic in a neighborhood N of E , it is shown that there is a polynomial P of degree n , such that $|f-P| < Ae^{-nB}$, where A and B are constants depending on N whose dependence on N is made explicit. An example shows that the results come close to being best possible. The third part of the paper deals with (weighted) uniform approximation by polynomials to functions analytic in certain closed, unbounded regions. For certain classes of such functions estimates are obtained for the rate of decrease of weight functions as $|z| \rightarrow \infty$ for which uniform approximation within ϵ is possible. The exact statements are too long to be reproduced here.
W. H. J. Fuchs.

Kalašnikov, M. D. On polynomials of best (quadratic) approximation at a given system of points. *Dokl. Akad. Nauk SSSR (N.S.)* 105 (1955), 634-636. (Russian)

The polynomial $T_n^N(x) = \frac{1}{2}x_0 + \sum_{k=1}^n (\alpha_k \cos kx + \beta_k \sin kx)$ with coefficients

$$\alpha_k = \frac{2}{N} \sum_{i=1}^N f(x_i) \cos kx_i, \quad \beta_k = \frac{2}{N} \sum_{i=1}^N f(x_i) \sin kx_i,$$

$x_i = 2\pi i/N$, $i=1, \dots, N$, $N > 2n$, minimizes the sum $\sum_{i=1}^N \{f(x_i) - T_n^N(x_i)\}^2$. The norm of $T_n^N(x)$, as a functional on the space \bar{C} of continuous periodic functions, is computed; if $N=m(2n+1)$, m an integer, then the norm is

$$2 \cos [\pi/2m - (n+\frac{1}{2})x] \log n/m\pi \sin (\pi/2m) + O(1).$$

The author also gives a similar expression for the norm of

$$T_{n,k}^N(x) = \frac{1}{2} \{T_n^N(x) + T_k^N(x + \frac{2k\pi}{2n+1})\}$$

and derives conditions under which $T_{n,k}^N(x) \rightarrow f(x)$ for all $f \in \bar{C}$, uniformly in m and x .
G. G. Lorentz.

Ibragimov, I. I. On best approximation by polynomials of a function whose s -th derivative has a discontinuity of the first kind. *Izv. Akad. Nauk Azerbaidžan. SSR.* 1953, no. 3, 19-48. (Russian. Azerbaijani summary)

Ibragimov, I. I. On best approximation in the mean of differentiable functions by polynomials. *Dokl. Akad. Nauk Azerbaidžan. SSR.* 9 (1953), 135-141. (Russian. Azerbaijani summary)

[The first paper was summarized in *Dokl. Akad. Nauk SSSR (N.S.)* 89 (1953), 973-975; 90 (1953), 13-15; MR 14, 1083]. The author discusses the topic described in the first title, where s is any positive number. The function that is approximated is defined on $[-1, 1]$ and the integral is the Riemann-Liouville integral with origin -1 . Consider the best uniform and best L^1 approximations to a given function by polynomials. It turns out that these quantities can be calculated asymptotically if one knows them for the functions $[a(x-c)+b|x-c|]|x-c|^\theta$, where $|c| < 1$, and a large part of the paper is devoted to calculating them in this case. The author also considers L^1 approximation when the s th derivative is the sum of a step function and an absolutely continuous function, and [second paper] when the s th derivative has either total variation at most 1 or L^1 -norm at most 1. The results are too complicated to reproduce here.
R. P. Boas, Jr.

Bredihina, E. A. Some estimates of best approximations of almost-periodic functions. *Dokl. Akad. Nauk SSSR (N.S.)* 103 (1955), 751-754. (Russian)

The author considers the best approximation of almost periodic functions $f(x)$ whose sequence of Fourier exponents, Λ_0 , is a sub-sequence of a given sequence

$$\Lambda = \{\lambda_0=0, \lambda_1=-\lambda_{-1}, \lambda_2=-\lambda_{-2}, \dots\}, \quad \lambda_k \rightarrow \infty,$$

by trigonometric polynomials $P_n(x) = \sum c_k e^{i\lambda_k x}$ ($\lambda_k \in \Lambda$, $|\lambda_k| < \mu$). It is proved that the best approximation by $P_n(x)$ to $f(x)$ is $O(\omega(1/\mu))$, $\omega(\delta)$ modulus of continuity of $f(x)$. This remains true, if the Fourier exponents of P_n are chosen from Λ_0 only.

In the case of real-valued gap series

$$\sum (a_k e^{i\lambda_k x} + b_k e^{-i\lambda_k x}), \quad \lambda_{m+1}/\lambda_m \geq c > 8,$$

it is proved that the best approximation by $P_n(x)$ is of the same order of magnitude as

$$\sum_{|\lambda_k| > \mu} (|a_k| + |b_k|)$$

and that it is asymptotically equal to this sum, if the gaps increase sufficiently rapidly.
W. H. J. Fuchs.

Freud, Géza. Ein Zusammenhang zwischen den Funktionenklassen $Lip \alpha$ und $Lip(\beta, \rho)$. Acta Sci. Math. Szeged 15 (1954), 260; Berichtigung 16 (1955), 28.

It is shown by reference to theorems of Szász and Zygmund that the functions in the class $Lip(\beta, \rho)$ ($0 < \rho^{-1} < \beta \leq 1$) are equivalent to functions belonging to the class $Lip \alpha$ for $0 < \alpha < \beta - \rho^{-1}$. The author observes in a subsequent note that the result is a special case of a theorem of Hardy and Littlewood [Math. Z. 28 (1928), 612-634] and a generalization of it due to Ya. L. Geronimus [Dokl. Akad. Nauk SSSR (N.S.) 88 (1953), 597-599; MR 14, 871]. G. Klein (Cambridge, Mass.).

Mergelyan, S. N. General metric criteria of completeness of a system of polynomials. Dokl. Akad. Nauk SSSR (N.S.) 105 (1955), 901-904. (Russian)

Let D be a bounded, simply connected domain in the z -plane and let S be a circle containing \bar{D} in its interior. Let $L^2(D)$ be the space of functions analytic in D and of finite L^2 -norm: $\|f\|^2 = \iint_D |f(z)|^2 ds < \infty$. Theorem. The polynomials are complete in $L^2(D)$, if there is a sequence of points $\{\zeta_n\}$ with the following properties: 1. The closure of the point set $\{\zeta_n\}$ contains the boundary of D . 2. To every n there is a δ_n such that for $\delta < \delta_n$ the point ζ_n can be reached from S along a curve λ of length L such that $\text{meas } D_\delta(\lambda) < \exp(-\exp(5L/\delta))$, where $D_\delta(\lambda)$ is the set of points in D whose distance from λ is less than δ . The proof uses Keldyś's results about Runge's method of pole-shifting [see Mergelyan, Uspehi Mat. Nauk (N.S.) 8 (1953), no. 4 (56), 3-63; MR 15, 411]. In the exponent $5L/\delta$ can not be replaced by $(L/\delta)^\gamma$, $\gamma < 1$. The author states the following conjecture: A necessary and sufficient condition for the polynomials to be complete in $L^2(D)$ is that for every boundary point ζ of D and every choice of $\varepsilon > 0$ there is a polynomial $P(z)$ satisfying $\|P\| < \varepsilon$ and $\sup_{|z-\zeta| \geq \varepsilon} |P(z)| > 1$. W. H. J. Fuchs (Ithaca, N. Y.).

Egorova, I. A. On a property of the roots of Jacobi polynomials. Leningrad. Gos. Ped. Inst. Uč. Zap. 89 (1953), 153-159. (Russian)

Let $x_k^{(n)} = \cos \theta_k^{(n)}$ ($k=1, \dots, n$; $n=1, 2, \dots$) represent the zeros of Jacobi polynomials with weight function $(1-x)^\alpha(1+x)^\beta$, $-\frac{1}{2} < \alpha < \frac{1}{2}$, $-\frac{1}{2} < \beta < \frac{1}{2}$, and let $N = n + \frac{1}{2}(\alpha + \beta + 1)$ and $N[\theta_{k+1}^{(n)} - \theta_k^{(n)}] = \pi + \alpha_k^{(n)}$. Then the author proves that $-A < \sum_{k=1}^n \alpha_k^{(n)} < A$, and for $-\frac{1}{2} < \alpha = \beta < \frac{1}{2}$, $\sum_{k=1}^n |\alpha_k^{(n)}| < 6A$, where A is a positive number. S. Kulik (Columbia, S. C.).

Walsh, Joseph L.; and Zedek, Mishaël. On generalized Tchebycheff polynomials. Proc. Nat. Acad. Sci. U.S.A. 42 (1956), 99-104.

We denote by n and s fixed integers, $1 < s < n$. Let us consider the class A of the polynomials of degree n having given leading terms $x^n + A_1 x^{n-1} + \dots + A_s x^{n-s}$, and let E be a set of at least $n-s-1$ points. The authors are concerned with the following generalization $T_n^s(z) \in A$ of the Tchebychev polynomials: $\max |T_n^s(z)|$, $z \in E$, is a minimum. As a generalization of certain concepts due to Fekete, Motzkin, and Walsh, they introduce the "infra polynomials" $I_n^s(z) \in A$, defined by the condition that there is no $p_n^s(z) \in A$ such that $|p_n^s(z)| < |I_n^s(z)|$, $z \in E$, except perhaps for common zeros of the two polynomials. As a generalization of a result of Fekete-von Neumann, $T_n^s(z)$ is computed in the special case when E consists of exactly $n-s+1$ points. As a refinement of a result of Fejér it is shown that if z_k denote $s+1$ zeros of $I_n^s(z)$ and φ_k the angles subtended by E at z_k , we have $\sum \varphi_k \geq \pi$.

Finally a certain "orthogonality" theorem and a "structure" theorem of Fekete (concerning the polynomials $I_n^0(z)$) is extended from $s=0$ to an arbitrary value of s . G. Szegő (Stanford, Calif.).

★ Pollaczek, Félix. Sur une généralisation des polynômes de Jacobi. Mémor. Sci. Math., no. 131. Gauthier-Villars, Paris, 1956, 55 pp. 1000 francs.

The polynomials $P_n(z)$ defined by the recursion $P_n(z) = (A_n z + B_n)P_{n-1}(z) - C_n P_{n-2}(z)$, $A_n \neq 0$, $n=1, 2, \dots$, are orthogonal with respect to a certain distribution provided $C_n/(A_{n-1}A_n) > 0$ for all n . The aim of this monograph is to study the case when A_n, B_n, C_n are rational functions of a fixed degree m in n :

$$A_n = \frac{p_1}{p_0}(n), \quad B_n = \frac{p_2}{p_0}(n), \quad C_n = \frac{p_3}{p_0}(n),$$

where $p_i(x) = \alpha_{0i}x^m + \alpha_{1i}x^{m-1} + \dots + \alpha_{mi}$; $i=0, \dots, 3$; $m \geq 1$, $\alpha_{00}=1$, $\alpha_{01} \neq 0$, $\alpha_{03} \neq 0$; moreover $p_0(x) \neq 0$, $p_1(x) \neq 0$ for $x=1, 2, 3, \dots$. For the generating function $g(x, z) = \sum_{n=0}^{\infty} x^n P_n(z)$ a differential equation of the form $\sum_{v=0}^m q_v(x, z) g^{(m-v)} = p_0(0)$ is obtained, where the differentiation is with respect to x and $q_v(x, z)$ has the form $x^{m-v}(a_{v0} - (a_{v1}z + a_{v2})x + a_{v3}x^2)$, $1 \leq v \leq m$;

$$q_0(x) = x^m(x^2 - 2zx + 1).$$

Thus the singularities are at $x=0$ and at the roots α, β of $x^2 - 2zx + 1 = 0$. Based on the discussion of the corresponding homogeneous equation the solution $g(x, z)$ is represented in terms of certain integrals taken in the sense of Hadamard. This leads to a representation of $P_n(z)$ as a contour integral. By a proper choice of the contour an asymptotic formula for $P_n(z)$, n large, is obtained whose main terms are of the form $n^A x^{-n} r(z) + n^B \beta^{-n} s(z)$, where $r(z)$ and $s(z)$ are certain fixed functions; here $z \neq \pm 1$. Also an explicit representation of $P_n(z)$ in terms of Hadamard integrals is given. Several interesting special cases are discussed in which the distribution function of the orthogonality is obtained. G. Szegő (Stanford, Calif.).

Laščenov, K. V. On a class of orthogonal polynomials. Leningrad. Gos. Ped. Inst. Uč. Zap. 89 (1953), 167-189. (Russian)

The polynomials $R_n^{(p,q)}(x) = \alpha_n x^n + \alpha_{n-2} x^{n-2} + \dots$ ($\alpha_n \neq 0$, $p > -1$, $q > -1$) orthogonal over the interval $[-1, 1]$ with respect to the weight function $(1-x^2)^p |x|^q$, are constant multiples of

$$\frac{P_m^{(p, (q-1)/2)}(2x^2-1)}{x P_m^{(p, (q+1)/2)}(2x^2-1)}, \quad n=2m, \\ x P_m^{(p, (q+1)/2)}(2x^2-1), \quad n=2m+1,$$

$P_j^{(\alpha, \beta)}(t)$ being the classical Jacobi polynomial of degree j with parameters α and β . This paper is a detailed derivation of the above fact, followed by a transformation to the polynomials $R_n^{(p,q)}(x)$ of the known recurrence formula, differential equation, bounds, convergence theorems, and distribution of zeros for the Jacobi polynomials [G. Szegő, Orthogonal polynomials, Amer. Math. Soc. Colloq. Publ., v. 23, New York, 1939; chap. 4; MR 1, 14]. A. E. Livingston.

Laščenov, K. V. On interpolation with the roots of orthogonal polynomials of weight $(1-x^2)^p |x|^q$. Leningrad. Gos. Ped. Inst. Uč. Zap. 89 (1953), 191-206. (Russian)

Theorem 14.4 of Szegő [Orthogonal polynomials, Amer. Math. Soc. Colloq. Publ., v. 23, New York, 1939; MR 1, 14] is modified so as to be applicable to Lagrange interpolation by means of the roots of the polynomials $R_n^{(p,q)}(x)$ defined in the preceding review.

A. E. Livingston (Seattle, Wash.).

*Doetsch, Gustav. *Teoria degli sviluppi asintotici dal punto di vista delle trasformazioni funzionali*. Consiglio Naz. Ricerche. Pubbl. Ist. Appl. Calcolo, no. 420. Casa Editrice Libreria Rosenberg & Sellier, Torino, 1954. 86 pp.

This paper contains an extensive and clear exposition (practically without proofs) of asymptotics based on functional transformations. Part I (pp. 1-13) gives general considerations on asymptotic expansions and on the goal of the paper. Suppose that a function $\Phi(x)$ belonging to a certain class of functions corresponds to a function $\varphi(y)$ of another class by means of the transformation $\mathfrak{L}(\Phi(x)) = \varphi(y)$. If the asymptotic behaviour of $\Phi(x)$ is given in the neighborhood of a singular point x_0 , under what conditions is it possible to determine the asymptotic behaviour of $\varphi(y)$ at the corresponding singular point y_0 ?

In part II (pp. 14-57) the author treats the Laplace and Mellin transforms, $\mathfrak{L}(F)$ and $\mathfrak{M}(F)$. The asymptotic behaviour of $F(t)$ in the neighborhood of the origin yields the asymptotic behaviour of $\mathfrak{L}(F)$ as $|s| \rightarrow \infty$ in the sector $-\frac{1}{2}\pi < \arg s < \frac{1}{2}\pi$. If $F(t)$ is analytic in a sector $\alpha < \arg t < \beta$, then the asymptotic behaviour of $\mathfrak{L}(F)$ is even determined in the sector $-\beta - \frac{1}{2}\pi < \arg s < -\alpha + \frac{1}{2}\pi$. (Examples: Error integral, cylinder functions, incomplete gamma function, Bessel functions.) The case in which the asymptotic expansion of $F(t)$ contains functions such as $\log t^{-1}$, $\log \log t^{-1}$, \dots , e^{-1/t^2} is also treated. In a section devoted to the Laplace problem of functions of large numbers, the author determines the asymptotic expansion of $\int_0^\infty e^{st} g(x) dx$ for large $|s|$; the coefficients occurring in this asymptotic expansion are determined by means of the theorem of Lagrange-Bürmann. The integral $\int_0^\infty \exp(ix^2) dx$ is treated by means of the method of steepest descent. The author examines the domain of validity of the asymptotic expansion of $\mathfrak{L}(F)$ in the case that the integration path is a broken line and he indicates how to deform the integration path in order to extend the domain of validity of the asymptotic expansion. Example: $\int_0^\infty \exp(ixx^q) dx$, where $q > 0$. The author determines the asymptotic behaviour of the Laplace and Mellin transforms not only at infinity, but also in the neighborhood of a finite singular point. Part III (pp. 58-83) is devoted to the corresponding problems for the inverse transformations of \mathfrak{L} and \mathfrak{M} .

J. G. van der Corput.

See also: Evgrafov, p. 749; Stiefel, p. 790.

Trigonometric Series and Integrals

Tureckil, A. H. On a function deviating least from zero. Belorussk. Gos. Univ. Uč. Zap. Ser. Fiz.-Mat. 19 (1954), 41-43. (Russian)

The author seeks to determine $M_n = \max \phi_n(x)$ where $\phi_n(x) = \sum_{k=1}^n b_k |\sin kx|$, with $b_k \geq 0$ and $\sum_{k=1}^n b_k = 1$. He finds $2/\pi \leq M_n \leq (2/\pi)(1+n^{-1})$, where the upper bound arises from $b_k = 2(n-k+1)n^{-1}(n+1)^{-1}$.

R. P. Boas, Jr. (Evanston, Ill.).

Džvaršėšvili, A. G. On the convergence of trigonometric series. Soobšč. Akad. Nauk Gruzin. SSR 15 (1954), 65-68. (Russian)

Let $\Delta a_n = a_n - a_{n+1}$ and $\Delta^{p+1} a_n = \Delta^p a_n - \Delta^p a_{n+1}$. A trigonometric series

$$T(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

is said to belong to the class V_p if the sequences $\{a_n\}$ and $\{b_n\}$ tend to zero and $|\Delta^p a_n| < M/n$, $|\Delta^p b_n| < M/n$, and a function f in $L(-\pi, \pi)$ is in V_p if its Fourier series $S(f)$ is. The following result is given as an extension of Hardy's Tauberian theorem: if $T(x) \in V_p$ ($p > 0$) is summable (C, α) ($\alpha > 0$), then $T(x)$ converges almost everywhere (a.e.) in $(-\pi, \pi)$; if $f \in V_p$, then $S(f)$ converges a.e. to $f(x)$. As a generalization of the theorem of Kolmogoroff and Seliverstoff the author states the following: if $T(x)$ is summable (C, α) ($\alpha > 0$) on $(-\pi, \pi)$ then $T(x)$ converges a.e. provided $\sum_{n=1}^{\infty} ((\Delta^p a_n)^2 + (\Delta^p b_n)^2) \log n$ converges. The corresponding results for $S(f)$ and its conjugate series $\tilde{S}(f)$ are also given.

G. Klein (Cambridge, Mass.).

Tandori, Károly. Über die Divergenz der Fourierreihen.

Acta Sci. Math. Szeged 15 (1954), 236-239.

A. G. Džvaršėšvili has shown [Soobšč. Akad. Nauk Gruzin. SSR 11 (1950), 403-407; MR 14, 635] that if a function $f(x) \in L$ vanishes in a closed set E in $[-\pi, \pi]$ then the Fourier series of f converges to zero at every point of density of E provided (*) $\sum_{k=1}^{\infty} \omega(f; \delta_k) < +\infty$, where $\{\delta_k\}$ are the intervals contiguous to E and $\omega(f; \delta_k)$ denotes the variation of f on δ_k . The author shows by construction that the theorem does not remain true if condition (*) is suppressed.

G. Klein (Cambridge, Mass.).

du Plessis, N. A note about functions in Lip α . Proc. Edinburgh Math. Soc. (2) 10 (1954), 100.

It is shown that a necessary and sufficient condition that a trigonometrical series $T(x)$ be the Fourier series of a function $f(x)$ in Lip α ($0 < \alpha < 1$) is that $\sigma_n - \sigma_m = O(n^{-\alpha})$ uniformly in $[0, 2\pi]$ for all $m > n$, where σ_n is the n th ($C, 1$) mean of $T(x)$.

G. Klein (Cambridge, Mass.).

Izumi, Shin-ichi, and Satō, Masako. Some trigonometrical series. XVII. Proc. Japan Acad. 31 (1955), 659-664.

Theorems on the γ th functional integral $f_\gamma(x)$ of a function $f(x)$ and its Fourier series. If $p > 1$, $p^{-1} < \alpha < 1$, and $f \in L^p$, then $|s_n(x, f_\alpha) - f_\alpha(x)| \leq A/n^{\alpha-1/p}$, where $s_n(x, f_\alpha)$ denotes the n th partial sum of the Fourier series of f_α at x . If also $\gamma > 0$ and $\alpha + \gamma < 1$, and $f \in \text{Lip}(\alpha, p)$, then

$$|s_n(x, f_\gamma) - f_\gamma(x)| \leq A/n^{\alpha-1/p+\gamma}.$$

W. W. Rogosinski (Newcastle-upon-Tyne).

Sokolov, I. G. The remainder term in the Fourier series of differentiable functions. L'vov. Gos. Univ. Uč. Zap. 29, Ser. Meh.-Mat. no. 6 (1954), 82-87. (Russian)

The author transforms the Lebesgue constants L_n [Zygmund, Trigonometrical series, Warsaw-Lwów, 1935, p. 172] into a form suitable for numerical calculation and tabulates L_n for $n=1(1)10, 5D$. He calculates numerically the coefficients in the asymptotic formula

$$L_n = a_1 \log(2n+1) + a_2 + a_3(2n+1)^{-2} + O(n^{-3}).$$

Let W_r be the class of functions of period 2π with r th derivative bounded by M ; let $H_n^{(r)} = \sup E_n(f)$ for $f \in W_r$, where E_n denotes best approximation by trigonometric

polynomials. Then [Ahiezer, Lectures on the theory of approximation, OGIz, Moscow-Leningrad, 1947, p. 207; MR 10, 33] $H_n^{(r)} = 4\pi^{-1}K_r(n+1)^{-r}M$, where K_r is expressed in terms of Bernoulli and Euler numbers. The author tabulates $4\pi^{-1}K_r$ for $r=1(1)10, 4D$.

R. P. Boas, Jr. (Evanston, Ill.).

Ul'yanov, P. L. On extension of functions. Dokl. Akad. Nauk SSSR (N.S.) 105 (1955), 913-915. (Russian)

Several theorems are announced, of which the following is typical. Let f be a periodic function on $[0, 2\pi]$ such that both f and \dot{f} , where

$$\dot{f}(x) = -\lim_{t \rightarrow 0} \frac{1}{2\pi} \int_0^{2\pi} [f(x+t) - f(x-t)] \cot \frac{t}{2} dt,$$

are continuous at every point of $[a, b] \subset [0, 2\pi]$. Then there is a function f_1 on $[0, 2\pi]$ agreeing with f on $[a, b]$ such that both f_1 and \dot{f}_1 are continuous everywhere on $[0, 2\pi]$. The general method of proof is indicated. E. Hewitt.

Kahane, Jean-Pierre. Sur quelques problèmes d'unicité et de prolongement, relatifs aux fonctions approchables par des sommes d'exponentielles. Ann. Inst. Fourier, Grenoble 5 (1953-1954), 39-130 (1955).

L'introduction des transformées de Fourier des fonctions moyenne-périodiques, introduites par Delsarte et dont la théorie est due à Schwartz, tout en permettant à l'auteur de retrouver d'une manière simple et rapide les résultats essentiels de cette théorie en la complétant sur certains points, lui donne aussi la possibilité de généraliser à ces fonctions plusieurs problèmes importants d'unicité et de prolongement. Dans un espace vectoriel topologique \mathcal{E} de fonctions f à valeurs complexes sur la droite, où la convergence est définie comme convergence uniforme sur tout segment, f est moyenne-périodique s'il existe au moins une mesure $\mu \in \mathcal{E}'$ (à support compact) non identiquement nulle telle que $f * \mu = 0$. La borne inférieure des longueurs des segments-supports de telles mesures est la moyenne-période de f . Pour chaque $\mu \in \mathcal{E}'$, $\mu \neq 0$, $f * \mu = 0$, posons $g = f * \mu = -f^+ * \mu$, où $f^+(x) = f(x)$ pour $x \leq 0$, $= 0$ pour $x > 0$ (définition analogue pour f^-), et soit $M(w) = \tau(\mu) = (2\pi)^{-1} \int_0^{2\pi} e^{-ixw} d\mu(x)$, $G(w) = \tau(g) = (2\pi)^{-1} \int_0^{2\pi} e^{-ixw} dg(x) dx$. La transformée de Fourier de f est définie par $\tau(f) = F(w) = G(w)/M(w)$. Cette fonction méromorphe (G et M sont des fonctions entières du type exponentiel) est indépendante de μ . Si $|f|$ ne croît pas trop vite, cette définition coïncide avec celle de Carleman. Le spectre de f est l'ensemble des pôles de F , comptés avec leur ordre de multiplicité. Une condition nécessaire et suffisante pour que l'exponentielle-monome $P(x)e^{i\lambda x}$, $P(x)$ polynôme de degré p , appartienne à $\tau(f)$ — le sous-espace de \mathcal{E} engendré par les translatées de f — est que λ soit un pôle de F d'ordre supérieur à p .

Voici quelques problèmes traités par l'auteur. Λ étant une suite de points (multiples) du plan complexe, déterminer l de façon que le sous-espace $\mathcal{E}(\Lambda) \subset \mathcal{E}$ engendré par les exponentielles-monomes $P(x)e^{i\lambda x}$ de spectre contenu dans Λ , soit une classe q.a. I_1 (quasi-analytique I_1), c'est-à-dire de façon que toute f de $\mathcal{E}(\Lambda)$ soit déterminée par ses valeurs sur un intervalle arbitraire de longueur l . Déterminer $I(\Lambda)$ de façon que $\mathcal{E}(\Lambda)$ soit q.a. $I(\alpha)$, c'est-à-dire pour que de $f \in \mathcal{E}(\Lambda)$ et $\int_0^x |f(x)| dx < I(\alpha)$, pour $0 < \alpha < x_0$, résulte que $f = 0$. Trouver des conditions (portant sur Λ et $\{M_n\}$) pour que de $f \in \mathcal{E}(\Lambda) \cap C(M_n)$, $f^{(n)}(0) = 0$ ($n \geq 0$) résulte que $f = 0$ (q.a. D). Sa solution du problème

q.a. I_1 généralise un théorème de Levinson (correspondant au cas où Λ est une suite d'entiers), l'auteur confirme aussi une hypothèse de L. Schwartz concernant la "moyenne-période attachée à Λ ". Le problème de la q.a. I_1 conduit naturellement au problème de prolongement en une fonction de $\mathcal{E}(\Lambda)$ d'une fonction définie sur un segment I . Le problème de q.a. $I(\alpha)$ pour les fonctions périodiques remonte à un théorème de Mandelbrojt; il a été généralisé par Lévine et Lifschitz, Lévine, Hirschman et Jenkins. Le problème de q.a. D a également été traité par le premier de ces auteurs. Les méthodes qu'utilise Kahane pour ces problèmes sont d'une grande variété et d'une originalité profonde. Le type même des théorèmes concernant la q.a. $I(\alpha)$, ou la q.a. D , dépend de la manière dont les éléments λ de Λ sont distribués. Ainsi, par exemple, lorsque les λ sont assez rares, ayant, tous, la partie imaginaire bornée, une condition nécessaire et suffisante de la q.a. $I(\alpha)$ de $\mathcal{E}(\Lambda)$ est que: $\liminf_{\lambda \rightarrow 0} (I(\alpha)/\lambda |f_0|) = 0$, f_0 étant une fonction de $\mathcal{E}(\Lambda)$ dont la transformée de Fourier est obtenue simplement à partir de Λ ; f_0 est "la plus petite" fonction de $\mathcal{E}(\Lambda)$, près de l'origine, non identiquement nulle. Dans le domaine complexe, où les théorèmes de Kahane sont nombreux, citons celui-ci: toute fonction moyenne-périodique de moyenne période L , analytique sur la droite réelle, est prolongeable analytiquement dans une bande horizontale B telle que tout segment de longueur L de la frontière de B contienne au moins un point singulier (à moins que cette fonction soit entière). En restant dans le domaine complexe il y a lieu de citer le fait suivant, établi par l'auteur, et qui lui sert dans ses problèmes de prolongement dans le domaine complexe: si $D(x)$ est une fonction entière de type exponentiel k , bornée sur la droite réelle, on a $\lim_{r \rightarrow \infty} (\log |D(re^{i\theta})|) r^{-1} = k |\sin \theta|$ pour presque toutes les valeurs de θ . Le dernier chapitre de ce travail contient des résultats difficiles à exprimer en quelques mots, et qui concernent des problèmes d'approximation et d'unicité dans le champ réel: problèmes généralisés des moments, de l'approximation polynomiale, de la quasi-analyticité généralisée. Ces résultats sont proches de ceux obtenus par Mandelbrojt à l'aide des séries adhérentes. En passant on trouve une extension intéressante d'un théorème de Mandelbrojt-MacLane sur les fonctions holomorphes dans une bande.

S. Mandelbrojt (Paris).

See also: Fourier, p. 698; Tveritin, p. 733.

Integral Transforms, Operational Calculus

★ Sneddon, I. N. Functional analysis. Handbuch der Physik, Bd. II, pp. 198-348. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1955. DM 88.00.

This exposition of functional analysis is addressed to theoretical physicists and others mainly interested in applying the theory. The major part of the article consists of a substantial summary of definitions and operational properties of linear integral transformations that are useful in solving boundary-value problems in differential equations. The survey of Laplace transformations includes sections on the bilateral, double, iterated and Laplace-Stieltjes transformations. The types of Fourier transforms treated are: exponential, sine, cosine, a modified sine-cosine transform pertaining to boundary conditions of the third type, finite sine and cosine transforms and their modifications, and multiple finite transforms. Hankel and finite Hankel transformations,

Legendre integral and Mellin transformations are included. Illustrative applications, usually to the solution of problems in partial differential equations, are given for many of the transformations. Approximate methods of computing integral transforms are surveyed. Brief tables of Laplace, Fourier exponential, sine and cosine transforms are given. These tables, together with the discussion of inverse transformations, convolution properties and other operational properties of the transformations, make the article adequate for use in solving a good variety of boundary-value problems. For the treatment of other problems the reader is provided with an up-to-date bibliography. Except for the bibliography, the material mentioned above does not depend on the treatment of integration, abstract spaces and Banach space, presented in the first twenty pages of the article. Hilbert space and the theory of distributions are discussed briefly in the final pages of the article.

R. V. Churchill.

Conolly, B. W. On integral transforms. Proc. Edinburgh Math. Soc. (2) 10 (1956), 125-128.

The author describes a very general formal process for obtaining new integral transforms from old ones. He applies this process to deduce

$$f(x) = \int_0^\infty K_x(\sqrt{xy})g(y)dy,$$

$$g(y) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} I_s(\sqrt{xy})f(x)dx$$

from the Laplace transformation and its complex inversion formula, and similarly obtains the inversions of a functional transformation involving Whittaker's confluent hypergeometric function.

A. Erdélyi.

Singh, Vikramaditya. Convergence theorems for a generalized Laplace integral. Math. Z. 64 (1955), 1-9 (1956).

The generalized Laplace integral is

$$f(s) = \int_0^\infty (st)^{m-1} e^{-st} W_{k,m}(st) d\alpha(t),$$

where $W_{k,m}$ is Whittaker's function. In generalization of known results on Laplace integrals, the author gives theorems on abscissae of convergence and regions of uniform convergence, Abelian and Tauberian theorems. Much of the reasoning is unintelligible to the reviewer. It seems to be assumed, but not proved, that the domain of convergence is a half-plane; an inequality stated as

$$|\alpha(t)| < t^{-2l-3/2} \quad (t \rightarrow 0),$$

and probably meaning $\alpha(t) = O(t^{-2l-3/2})$ as $t \rightarrow +0$, seems to be used for all $t > 0$; the fact that a real integral (not its absolute value) is bounded above is accepted as proof of convergence, this inequality is said to be extended to complex values of s by analytic continuation; and so on.

A. Erdélyi (Pasadena, Calif.).

Toscano, Letterio. Nuove regole di calcolo simbolico. Boll. Un. Mat. Ital. (3) 10 (1955), 541-543.

If $f(t) \supset \phi(P)$, the author proves that

$$t^{n(1-u)} (t^u D_t)^n f(t) \supset (-1)^n (p^u D_p)^{n-1} p^{n-(n-1)u} D_p \phi(p),$$

where $D_x = d/dx$, and notes that this rule can be extended to Meijer transforms. [Reviewer's remark: the formal rule holds for all integral transforms whose kernel depends on the product of the variables.]

A. Erdélyi.

See also: Doetsch, p. 731; Toscano, p. 733.

Special Functions

Tveritin, A. N. On a class of trigonometric series. Dnepropetrov. Gos. Univ. Nauč. Zap. 41 (1953), 121-136. (Russian)

The author obtains some properties of the functions

$$U(a, x) = \sum_{n=1}^\infty (n+a)^{-1} \sin nx,$$

$$V(a, x) = \sum_{n=1}^\infty (n+a)^{-1} \cos nx.$$

For $a > -1$ they are represented by integral formulas that are special cases of those given previously [Dokl. Akad. Nauk SSSR (N.S.) 61 (1948), 985-988; MR 10, 187]. $U(a+1, x)$ and $V(a+1, x)$ are expressible in terms of $U(a, x)$ and $V(a, x)$. We have the differential equations (for fixed a)

$$U'(a, x) = -\frac{1}{2} - aV(a, x), \quad V''(a, x) = \frac{1}{2} \csc^2 \frac{1}{2}x + aU'(a, x),$$

$$U''(a, x) + a^2 U(a, x) = \frac{1}{2} a \cot \frac{1}{2}x,$$

$$V''(a, x) + a^2 V(a, x) = \frac{1}{2} \csc^2 \frac{1}{2}x - \frac{1}{2}a,$$

from which further integral representations are obtained. As $x \rightarrow 0+$,

$$U(a, x) = \frac{1}{2}\pi + ax \log x + k_1 x + k_2 x^2 + k_3 x^3 \log x + k_4 x^3 + o(x^3)$$

with explicit values for the k 's; hence the asymptotic form of $V(a, x)$ can be obtained. The graphs of the functions are discussed both for varying x and varying a . When a is rational, and in particular when it is a positive integer, U and V are expressed as explicit finite sums of elementary functions. Finally there is a brief discussion of the "remainders" in which the sums extend over $m \leq n < \infty$.

R. P. Boas, Jr. (Evanston, Ill.).

Toscano, Letterio. Sul complemento della funzione gamma incompleta nel calcolo simbolico. Boll. Un. Mat. Ital. (3) 10 (1955), 484-488.

Using the known operational counter-image of $\Gamma(a, p)$ in combination with the convolution formula, the author obtains the operational counter-image of $p e^{2p} \Gamma(a, p) \Gamma(b, p)$ as a combination of hypergeometric series, and lists various special cases of his result.

A. Erdélyi (Pasadena, Calif.).

Toscano, Letterio. Differenze finite e derivate dei polinomi di Iacobi. Matematiche, Catania 10 (1955), 44-56.

The m th difference with respect to α of the Jacobi polynomial $P_n^{(\alpha, \beta)}(x)$ is $[(x+1)/2]^m P_{n-m}^{(\alpha+m, \beta+m)}(x)$, and the m th backward difference, the m th differences with respect to β , and the m th derivative with respect to x can similarly be expressed. The author uses the known expression of finite differences in terms of derivatives, and vice versa, to obtain expressions for the derivatives of Jacobi polynomials with respect to α and β , and differences with respect to x . He repeats the process on other families of functions, notably ultraspherical and Laguerre polynomials and some others.

A. Erdélyi.

Toscano, Letterio. Osservazioni, confronti e complementi su particolari polinomi ipergeometrici. Matematiche, Catania 10 (1955), 121-133.

Halphen, Etienne. Les fonctions factorielles. Publ. Inst. Statist. Univ. Paris 4 (1955), 21-37.

The author investigates

$$cf_n(x) = {}_1F_1\left(\alpha; \frac{1}{2}; \frac{x^2}{4}\right), \quad sf_n(x) = x {}_1F_1\left(\alpha + \frac{1}{2}; \frac{1}{2}; \frac{x^2}{4}\right),$$

and several other functions closely related to parabolic cylinder functions, and obtains recurrence and differentiation formulas, integral representations, asymptotic expansions, etc. Most of these are known. *A. Erdélyi.*

Robin, Louis. *Développements asymptotiques des fonctions associées de Legendre, $P_n^m(\mu)$ et $Q_n^m(\mu)$, pour $|n| \rightarrow \infty, |\mu \pm 1| \rightarrow 0$.* C. R. Acad. Sci. Paris 242 (1956), 868-870.

The author extends work of Macdonald [Proc. London Math. Soc. (2) 13 (1914), 220-221] to obtain expressions asymptotic to $Q_n^m(\mu)$ when $|n| \rightarrow \infty$ and $|\mu - 1| \rightarrow 0$, and to $P_n^m(\mu)$ and $Q_n^m(\mu)$ when $|n| \rightarrow \infty$ and $|\mu + 1| \rightarrow 0$. The formulas involve cylinder functions of orders $m, m-1, m-2$, and $m-3$. The error terms are of order $(1 \pm \mu)^2$. In formula (4) for Y_n read Y_m . *N. D. Kazarinoff.*

Gatteschi, Luigi. *Sulla rappresentazione asintotica delle funzioni di Bessel di uguale ordine ed argomento.* Boll. Un. Mat. Ital. (3) 10 (1955), 531-536.

In the first part of this note the author rectifies an error in an earlier paper [Ann. Mat. Pura Appl. (4) 38 (1955), 267-280; MR 17, 34]; the result of that paper is not affected. In the second part of the present note he proves

$$Y_\nu(\nu) = -2^{-2/3} 3^{1/3} \pi^{-1} \nu^{-1/3} \Gamma(\frac{1}{3}) + o^*,$$

where $|o^*| \leq 0.252/(\nu\pi)$ for $\nu \geq 1$, and deduces

$$|J_\nu(\nu x)|, |Y_\nu(\nu x)| < \frac{1}{x^2} \left(\frac{3.841}{\pi \nu^2} + \frac{0.252}{\pi \nu} \right)$$

for $x > 1, \nu > 0$.

A. Erdélyi (Pasadena, Calif.).

See also: Toscano, p. 716; Ascoli, p. 740.

Ordinary Differential Equations

Künzi, Hans-P. *Sur un théorème de M. J. Malmquist.* C. R. Acad. Sci. Paris 242 (1956), 866-868.

Using Nevanlinna's theory of value distribution, the author gives a short and elegant proof of the following classical theorem of Malmquist: If the differential equation $dw/dz = f(z, w)$ does not reduce to a Riccati equation, then any finitely-valued solution must be an algebraic function. *Z. Nehari (Pittsburgh, Pa.).*

Wintner, Aurel. *On a theorem of Painlevé.* Arch. Math. 6 (1955), 439-441.

If w is a solution of the differential equation $dw/dz = f(z, w)$ then, according to a classical theorem of Painlevé, a singularity z_0 of $w(z)$ at which $w(z)$ takes a finite value, say w_0 , must correspond to a singularity (z_0, w_0) of the function $f(z, w)$. In the present paper, this theorem is generalized to the case of a system

$$dw/dz = f_\nu(z, w_1, \dots, w_n), \quad \nu = 1, \dots, n.$$

Z. Nehari (Pittsburgh, Pa.).

Sibuya, Yasutaka. *Sur les points singuliers d'une équation différentielle ordinaire du premier ordre.* Proc. Japan Acad. 31 (1955), 41-44.

The author considers the equation

$$(1) \quad xy' = y(\lambda + yf(x, y)),$$

where f is analytic for $x=0, y=0$. Siegel has shown [Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. Math.-Phys.-Chem. Abt. 1952, 21-30; MR 15, 222] that (1) has a formal solution

$$(2) \quad y = z[1 + \sum \delta_{mn} x^m z^n]$$

$$(m=0, 1, \dots; n=1, 2, \dots; z=cx^2)$$

which converges provided λ is irrational and satisfies certain inequalities. Dulac [J. Ecole Polytech. (2) 9 (1904), 1-125] has given an example for which (2) diverges. In this paper further examples are given for divergence, in particular ones of the form $xy' = \lambda y + y^2 + axy^3/(1-x)$.

W. Kaplan (Ann Arbor, Mich.).

★Bellman, R. *Teoriya ustoičivosti rešenii differentsial'nyh uravnenii.* [Stability theory of differential equations.] Translated by A. D. Myškis. Izdat. Inostran. Lit., Moscow, 1954. 216 pp. 10.85 rubles.

Translation of the author's Stability theory of differential equations [McGraw-Hill, New York, 1953; MR 15, 794].

★Rapoport, I. M. *O nekotorykh asimptotičeskikh metodah v teorii differentsial'nyh uravnenii.* [On some asymptotic methods in the theory of differential equations.] Izdat. Akad. Nauk Ukrain. SSR, Kiev, 1954. 292 pp. 19.35 rubles.

Questions of asymptotic behavior of the solutions of systems of ordinary linear differential equations, as either the independent variable, or a parameter, increase indefinitely, have been repeatedly discussed in the literature. The present book studies such questions for a few types of systems of linear differential equations. In the chosen frame, however, the presentation is consistent and complete; it is uniform in method and exhaustive in results. Practically one method is used throughout the book. Each section emphasizes only a few final theorems where all hypotheses and most notations are explicitly summarized, so that the text is self-explanatory. All this makes the book easy to read as well as suitable for reference.

Chapter I: Asymptotic behavior of the solutions of linear differential and difference equations. In §§ 1, 2 the author lays down the main tool which he then uses throughout the book, namely the reduction of systems of first-order ordinary differential equations to L -diagonal form. Here is a typical theorem: Given the real system

$$(1) \quad dx/dt = A(t)x, \quad t \geq 0,$$

$x = (x_1, \dots, x_n)$, $A(t) = \|a_{ij}(t)\|$ is an $n \times n$ matrix whose elements a_{ij} have first derivatives summable in $[0, \infty)$ [i.e., $a_{ij}' \in L[0, \infty)$] and the limit matrix $A(\infty)$ does not have multiple characteristic roots, then there exists a substitution $x = B(t)y$, $t \geq \tau$, τ sufficiently large, which brings (1) into

$$(2) \quad dy/dt = [W(t) + C(t)]y$$

[L -diagonal form], where $W(t)$ is a diagonal matrix whose diagonal elements $w_i(t)$ are the characteristic roots of the matrix $A(t)$, and $C(t) = \|c_{ij}(t)\|$ with $c_{ij}(t) \in L[0, +\infty)$. This theorem and more general ones can be retraced in L. Cesari [Ann. Scuola Norm. Sup. Pisa (2) 9 (1940), 163-186; Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. 11 (1940), 633-695; MR 3, 41.8, 208] for questions of stability. The method was successively used by N. Levinson [Duke

Math. J. 15 (1948), 111-126; MR 9, 509] for questions of behavior of the solutions [cf. R. Bellman, Stability theory of differential equations, McGraw-Hill, New York, 1953, Ch. 2; MR 15, 794 (Ch. 2)]. In § 3 extensions are discussed for the case where $A(\infty)$ has multiple characteristic roots. In all cases asymptotic representations of the solutions are given and their construction in terms of L -diagonal form is simple. In § 4 the method is applied to systems containing a parameter s , namely

$$(3) \quad dx/dt = [sA(t) + A_0(t) + s^{-1}R(t, s)]x, \quad t \geq 0,$$

where $A = \|a_{ij}\|$, $A_0 = \|a_{ij0}\|$, $R = \|r_{ij}\|$, and $a_{ij}, r_{ij} \in L[0, +\infty)$, $a_{ij0} \in L[0, +\infty)$, $\int_0^\infty |r_{ij}| dt < M < +\infty$, $i, j = 1, \dots, n$, $s \geq s_0$. Asymptotic formulas for the solutions of system (3) are given in the form

$$x_i = [b_{ij}(t) + s^{-1}\beta_{ij}(t, s)] \exp \int_0^t w_i(t) dt$$

($i=1, \dots, n$; $j=1, \dots, n$), and the functions b, β conveniently characterized. § 5 concerns difference systems of the form $y_{i, s+1} = w_i(y) + \sum_{j=1}^n c_{ij}(s)y_j$, where $s = s_0 + n$, $n=1, 2, \dots$.

Chapter II: Asymptotic behavior of the solutions of a self-adjoint linear differential equation. The $2n$ th-order real equation

$$(4) \quad q_0(t)x - \frac{d}{dt} \left[q_1(t) \frac{dx}{dt} - \frac{d}{dt} \left[\dots \frac{d}{dt} \left[q_n(t) \frac{d^n x}{dt^n} \right] \dots \right] \right] = \lambda p(t)x, \quad t \geq 0,$$

is studied with $q_0(t), \dots, q_{n-1}(t) \in L[0, +\infty)$, and real unknown functions $x(t)$ are sought which are continuous with $x', \dots, x^{(n-1)}, q_n x^{(n-1)} - (q_n x^{(n)})'$, etc., and the continuity of the functions $q_i(t)$ is not actually required. By reduction of (4) to a system in L -diagonal form, results are obtained concerning the behavior of the solutions of (4) as $t \rightarrow +\infty$ (§§ 1, 2). Here is a typical theorem: If (a) $q_i \in L[0, +\infty)$, $i=0, \dots, n-1$, $q_n^{-1}, p \in L(0, \tau)$, $q_n^{-2} q_n' \in L(\tau, \infty)$, $p' \in L(\tau, \infty)$;

(b) $q_n^{-1}(\infty) p(\infty) > 0$, for all τ sufficiently large, then, for $\lambda > 0$, (4) has $2n$ linearly independent solutions whose behavior as $t \rightarrow +\infty$ is given by the formulas

$$x_i = e^{s p(t)} \cos \alpha_i [p(\infty)/q_n(\infty)] [f_i(s) \cos(s p(t)) + g_i(s) \sin(s p(t))],$$

where

$$s = |\lambda|^{1/2n}, \quad \varphi(t) = \int_t^\infty [p(t) q_n^{-1}(t)]^{1/2n} dt,$$

and the other expressions are characterized. Analogous formulas hold for the derivatives of the x_i 's, and for $\lambda < 0$. In § 2 the cases $q_0^{-1}(t)p(t) \rightarrow 0$, $q_0(t)p^{-1}(t) \rightarrow 0$ as $t \rightarrow +\infty$, are studied. In § 3 the behavior of the solutions of (4) as $\lambda \rightarrow \infty$ is characterized in a form similar to the one above. No question concerning turning points are considered. The discussion concerns only the real field.

Chapter III: Asymptotic properties of the eigenvalues and eigenfunctions of the regular self-adjoint boundary problem. Equation (4) is considered with self-adjoint boundary conditions of the type

$$(5_1) \quad \sin a_K D_{K-1} x + \cos a_K D_{2n-K} x = 0 \text{ at } t=0,$$

$$(5_2) \quad \sin b_K D_{K-1} x + \cos b_K D_{2n-K} x = 0 \text{ at } t=T,$$

where $D_0 x = x$, $D_K x = d^K x / dt^K$, $K=1, 2, \dots, n-1$, $D_n x = q_n(t) d^n x / dt^n$; $D_K x = q_{2n-K}(t) d^{2n-K} x / dt^{2n-K} - d D_{K-1} x / dt$, $K=n+1, \dots, 2n-1$, and where $T > 0$ is a given number. In §§ 1 and 2 general theorems are given concerning the existence and

properties of solutions satisfying the n conditions (5₁). In § 3 applications are made to the behavior of the eigenvalues and eigenfunctions of (4) with conditions (5) as $T \rightarrow +\infty$. Here is a typical theorem: If (a) holds, (c) $p(t) > 0$, $q_n^{-1}(\infty) > 0$, then for $\lambda > 0$, (4) has eigenvalues $s_n(T)$ and normalized eigenfunctions $\zeta_n(t, T)$, $0 \leq t \leq T$, such that, as $T \rightarrow +\infty$, we have

$$s_{n+1}(T) - s_n(T) = [p(\infty) q_n^{-1}(\infty)]^{1/2n} [\pi + o(1)] / T$$

and

$$\zeta_n(t, T) = 2^{1/2} T^{-1/2} [\omega(t, \lambda_n) + o(1)] [p(\infty)]^{-1/2} q_n^{-1}(s_n),$$

where $s = |\lambda|^{1/2n}$, and the functions ω, ϱ are characterized. The case $\lambda < 0$ is also studied. Finally, the boundary problem in $[0, \infty)$ is studied with conditions (5₁) at $t=0$ and $x(t)$ summable in $[0, \infty)$ with weight $p(t)$ [i.e. $x \in L_p[0, \infty)$]. Asymptotic evaluations are given for the eigenvalues and normalized eigenfunctions $\Xi_n(t)$. In § 4 the cases $q_0^{-1}(t)p(t) \rightarrow 0$, and $q_0(t)p^{-1}(t) \rightarrow 0$ as $t \rightarrow \infty$, are studied.

Chapter IV: The singular self-adjoint boundary problem. The boundary problem in $[0, \infty)$ for equation (4) mentioned above is further studied. A generalized Fourier transform is given leading to a Parseval identity and expansion theorems for arbitrary functions $f(t)$, $0 \leq t < +\infty$, with $f(t) \in L_p^2(0, \infty)$ are proved.

Chapter V: Stability of solutions of linear differential equations. The stability in the sense of Lyapunov is discussed for systems reducible to L -diagonal form. In particular, systems (1) are considered with periodic coefficients, i.e., $a_{ij}(t+T) = a_{ij}(t)$. Extensions of the results are discussed for systems (1) with $a_{ij}(t+T) - a_{ij}(t) \in L[0, \infty)$. Systems of the form $d^2 x / dt^2 = P(t)x$, $P(t+T) = P(t)$ are considered in detail. Also difference systems are considered as in Ch. I.

A bibliography of 340 papers is added. It is regrettable that very little reference is made in the body of the book to this extensive list. A few typographic errors have been detected which, however, do not disturb the reading.

L. Cesari (Lafayette, Ind.).

Koval', P. I. On stability of solutions of systems of difference equations. Dokl. Akad. Nauk SSSR (N.S.) 103 (1955), 549-551. (Russian)

Under consideration is a system of vector-matrix equations (1) $x_{s+1} = A_s x_s + b_s$ ($s=1, 2, \dots$). It is said that the solutions of (1) are asymptotically stable (a.s.), if for every solution x_s of the homogeneous system (1⁰) one has $\|x_s\| \rightarrow 0$ (for $s \rightarrow \infty$); (L) (Liapounoff) stability occurs when $\|x_s\| \leq c < \infty$ for every solution of (1⁰); (I) (instability) occurs when for some solution of (1⁰) one has $\limsup \|x_s\| = \infty$. Some of the results are as follows. If $\lim A_s = A$ exists and all the characteristic numbers (c.n.'s) of A are of modulus < 1 , then the solutions of (1) are a.s.; if amongst the c.n.'s of A there are some of modulus > 1 , the solutions of (1) are (I). If $\sum \|A_{s+1} - A_s\| < \infty$ and A has no multiple c.n.'s and the c.n.'s of the A_s are of modulus $\leq 1 + q_s$, where $\sum q_s < \infty$ ($q_s \geq 0$), then there is stability by (L). The results are similar to those of I. M. Rapoport [see pp. 58-63 of the book reviewed above], but under somewhat lighter hypotheses. W. J. Trjitzinsky.

* Mitropol'skiĭ, Yu. A. Nestacionarnye processy v nelineynyh kolebatel'nyh sistemah. [Transient processes in nonlinear oscillatory systems.] Izdat. Akad. Nauk Ukraĭn. SSR, Kiev, 1955. 283 pp. 12.55 rubles. This work is primarily concerned with the asymptotic

solution of the equation

$$(1) \quad \frac{d}{dt} \{m(\tau)\dot{x}\} + c(\tau)x = \varepsilon F(\theta, \tau, x, \dot{x}, \varepsilon)$$

on which Soviet writers, and notably the author, have spent a good deal of effort. In this system ε is small and $\tau = \varepsilon t$ is a slowly varying time — slow, for instance, relative to certain autonomous or impressed frequencies. Here the usual constants in research on quasi-linear systems, notably the mass and spring constant, are made functions of t through τ , and θ is an angle such that $\dot{\theta} = \nu(\tau)$. It is supposed that F has period 2π in θ and is indefinitely differentiable and often is a polynomial. As for τ , one restricts it to an interval $[0, T]$ and assumes that $m(\tau)$, $c(\tau)$ are strictly positive in that interval.

Let $m = m(0)$, $c = c(0)$, $\omega = (c/m)^{1/2}$. To cover even cases of resonance take as generating solution

$$(2) \quad x = a \cos(s\varphi + \psi), \quad \varphi = \theta/\tau,$$

where r/s is an irreducible fraction and r, s are not too large. The solution of the given equation is then taken as

(3) $x = a \cos(s\varphi + \psi) + \varepsilon u_1(\tau, a, \theta, s\varphi + \psi) + \varepsilon^2 u_2(\dots) + \dots$, where the u_k have periods 2π in θ and $s\varphi + \psi$. As for a, ψ they satisfy a pair of equations

$$(4) \quad \begin{cases} \dot{a} = \varepsilon A_1(\tau, a, \psi) + \varepsilon^2 A_2(\dots) + \dots, \\ \dot{\psi} = \theta(\tau) - \frac{s}{\tau} \nu(\theta) + \varepsilon B_1(\dots) + \varepsilon^2 B_2(\dots) + \dots. \end{cases}$$

Thus the solution of (1) is reduced to that of the simpler system (4). However, as is well known, the resulting series solutions need not be convergent and are only usable to obtain approximations. If one stops in (4) with terms in ε^m , then one obtains profitably (3) to terms in ε^{m-1} . Actually these approximations are not unique and they are made definite by imposing that the first harmonic should not be affected by the approximations. This is expressed analytically by

$$\int_0^{2\pi} u_h(\tau, a, \theta, \sigma) \frac{\sin \sigma}{\cos \sigma} d\sigma = 0 \quad (h=1, 2, 3, \dots).$$

Upon actually carrying (3) to ε^m there is obtained what is referred to as the "improved m th approximation". Very explicit formulas are given by the author for the first two approximations (formulas up to A_2, B_2 inclusive).

The preceding remarks will give an idea of the scope of the method developed in this book. Regarding its four chapters one may say this. The first is devoted to a large collection of examples, mainly from physics and engineering. The simplest is the pendulum with slowly varying arm. In the second chapter the method is developed and then applied to many special cases. Even details of numerical calculations in some cases are dealt with at length. In Ch. 3 the same method is applied to systems with N degrees of freedom, with the restriction, however, that there are N normal frequencies, only one of which varies slowly. This makes the system amenable to the method developed in Ch. 2. Again many detailed practical applications are considered. In the fourth and last chapter the method receives a partial mathematical justification. Under certain restrictive conditions it is shown that for a very large time the m th approximate solutions truly approximate as desired.

The careful and explicit details of computation with which this book abounds should make it particularly valuable to applied mathematicians. S. Lefschetz.

Kononenko, V. O. On oscillations in nonlinear systems with many degrees of freedom. Dokl. Akad. Nauk SSSR (N.S.) 105 (1955), 664-667. (Russian)
The system considered is of type

$$\dot{x} = Q(\theta, \tau)x + \varepsilon V(x, \theta, \tau, \varepsilon)$$

where x, V are n -vectors, V is of class C^∞ in its arguments other than ε for ε small, $\tau = \varepsilon t$, $\dot{\theta} = \nu(\tau)$, and V, Q have the period 2π in θ . The attack is closely related to the general argument of Mitropolskii [see the book reviewed above]. As in this book τ is a "slowly varying time", and dependence upon it indicates slow variation in time. The general method consists in assuming first τ constant and (explicit) ε zero, solving the adjoint system to the resulting linear system and proceeding from there more or less as in the book just quoted. S. Lefschetz (Mexico, D.F.).

Kazarinoff, N. D., and McKelvey, R. Asymptotic solution of differential equations in a domain containing a regular singular point. Canad. J. Math. 8 (1956), 97-104.

The differential equation considered is

$$(*) \quad z^2 \frac{d^2 u}{dz^2} - \lambda^2 P(z, \lambda) u = 0,$$

where $P(z, \lambda)$ is single-valued and analytic in the complex parameter λ at infinity and in z throughout a bounded, closed, simply connected domain D of the complex z -plane containing $z=0$. Under the assumption that $P(z, \lambda) \neq 0$ for z in D and $|\lambda|$ sufficiently large the authors obtain series in descending powers of λ which are formal solutions of (*), and whose partial sums provide asymptotic representations of a fundamental set of solutions of this equation. In conclusion, the general method is applied to the determination of an asymptotic expansion of the Whittaker function $M_{2am+b,m}(z)$, valid for a, b, z bounded while $\Re(m) \geq 0$ and $|m|$ is large. W. T. Reid.

Kazarinoff, Nicholas D. Asymptotic solution with respect to a parameter of a differential equation having an irregular singular point. Proc. Amer. Math. Soc. 7 (1956), 62-69.

The differential equation treated specifically is

$$(*) \quad \frac{d^2 u}{dx^2} - \lambda^2 Q(x, \lambda) u = 0,$$

to which a more general equation given initially by the author may be reduced by a suitable change of variables. It is assumed that for some real $\nu > 2$ the function $x^\nu Q(x, \lambda)$ is single-valued and analytic in x throughout a bounded region R of the complex x -plane containing the origin and in the complex parameter λ at infinity; moreover, it is required that $x^\nu Q(x, \infty) \neq 0$ in R . There is introduced a function $\xi(x, \lambda)$ that is determined by the coefficients of the Taylor expansion of $Q(x, \lambda)$ about $\lambda = \infty$, and which is in general multiple-valued in R , and the author shows that whenever R possesses a certain property that is phrased in terms of the behavior of $\xi(x, \lambda)$ then there exist polynomials

$$A_{\pm n}(x, \lambda) = \sum_{j=0}^n \alpha_{\pm j}(x) \lambda^j \quad (n=0, 1, \dots),$$

in $1/\lambda$ such that $e^{\pm \xi} A_{\pm n}(x, \lambda)$ are approximate solutions of (*) with error $e^{\pm \xi} O(\lambda^{-n-1})$. Details requisite for precise statement of results are too tedious to be presented here; the analysis has the general structure of that employed

by the author and McKelvey in the case of a regular singular point [see the paper reviewed above].

W. T. Reid (Evanston, Ill.).

Sibirskii, K. S. On conditions for the presence of a center and a focus. *Kišinev. Gos. Univ. Uč. Zap.* 11 (1954), 115-117. (Russian)

Saharnikov [Prikl. Mat. Meh. 12 (1948), 669-670; MR 10, 377] reduced the conditions for a center of

$$(1) \quad \frac{dy}{dx} = -\frac{x+ax^2+(2b+\alpha)xy+cy^2}{y+bx^2+(2c+\beta)xy+dy^2}$$

to certain 6 algebraic relations between the coefficients. The sixth however is indirect and in terms of the coefficients of a certain linear transform of (1). The author simplifies this system and reduces it to three direct relations.

S. Lefschetz (Mexico, D.F.).

Zubova, A. F. Investigation of the question of oscillations and stability of solutions of an equation of second order. *Dokl. Akad. Nauk SSSR (N.S.)* 105 (1955), 14-17. (Russian)

Criteria for the boundedness of the solutions of $u''+p(x)u=0$, with $p(x)$ continuous and of period $\omega>0$, are found by determining the numbers of the negative eigen-values of $u''+(p(x)+\lambda)u=0$ with periodic and with semi-periodic boundary conditions. Use of the Ritz method together with a theorem of Jacobi relates these numbers to the changes in sign in certain sequences of determinants, assumed not to vanish. The case $u''+(q_0+q_1 \cos x)u=0$ is worked in detail, using trigonometric orthonormal systems in the Ritz method. Numerical bounds for q_0 are given, relating to stability or oscillatory character in the cases

$$p(x)=q_0+\cos x, q_0+4 \cos x, q_0+\cos x+\cos 2x.$$

F. V. Atkinson (Canberra).

Petropavlovskaya, R. V. On oscillation of solutions of the equation $u''+p(x)u=0$. *Dokl. Akad. Nauk SSSR (N.S.)* 105 (1955), 29-31. (Russian)

Let $p(x)$ be continuous and bounded from below on $a \leq x < \infty$. Then a sufficient condition for the solutions of $u''+p(x)u=0$ to be oscillatory is that $\int_a^\infty p(t)dt$ should not tend as $x \rightarrow \infty$ to either a finite or an infinite limit; the more restrictive condition that $\int_a^\infty p(t)dt$ should not have a mean-value as $x \rightarrow \infty$ is contained in a result of P. Hartman [Amer. J. Math. 74 (1952), 389-400, criterion III; MR 14, 50, where another result is cited], who on the other hand weakened the half-boundedness condition. Next are three sets of sufficient conditions for non-oscillation. One of these requires the existence of a function $\phi(x)$ such that either $0 \leq \int_a^x p(t)dt \leq -\phi'$, or $-\phi' \leq \int_a^x p(t)dt \leq 0$, while another requires that

$$|C + \int_a^x (p + \phi^2)dt| \leq \phi.$$

Lastly there is the result that if $u''+p_r(x)u=0$, $r=1, 2$, both non-oscillatory, then so is

$$u''+(\alpha p_1(x)+(1-\alpha)p_2(x))u=0,$$

for $0 < \alpha < 1$. Several corollaries are noted. There are no proofs; these may, the reviewer notes, be based on the Riccati-equation method, used by Hartman (loc. cit.).

F. V. Atkinson (Canberra).

Dingle, R. B. The method of comparison equations in the solution of linear second-order differential equations (generalized W. K. B. method). *Appl. Sci. Res. B.* 5 (1956), 345-367.

The author discusses the WKB method and Langer's method for the asymptotic solution of ordinary differential equations, gives 4D tables with second differences of the functions $x^{\frac{1}{2}}[J_{-\frac{1}{2}}(x) \pm J_{\frac{1}{2}}(x)]$ for $x=.05(.05)10$ and $x^{\frac{1}{2}}[I_{-\frac{1}{2}}(x) \pm I_{\frac{1}{2}}(x)]$ for $x=.05(.05)3$, with 7D values of the first seven zeros of each of the first two functions, and discusses the calculation of correction terms to the Langer solutions. He also describes the general comparison technique and lists those differential equations suitable as comparison equations.

A. Erdélyi.

Bellman, Richard. Boundedness of the solutions of second order linear differential equations. *Duke Math. J.* 22 (1955), 511-513.

All solutions of $u''+p(t)u=0$ are bounded as $t \rightarrow +\infty$, provided that $p(t) \geq c > 0$, $p'(t)$ is continuous, and $\int_{-\infty}^{\infty} |p'(t)|dt < \infty$. The author observes that this is included in a result of L. A. Gusarov [Moskov. Gos. Univ. Uč. Zap. 165, Mat. 7 (1954), 223-237; MR 16, 477], but gives a very simple proof.

G. E. H. Reuter (Manchester).

Greguš, Michal. On some properties of the solutions of a homogeneous linear differential equation of the third order. *Mat.-Fyz. Časopis. Slovensk. Akad. Vied* 5 (1955), 73-85. (Slovak. Russian summary)

The author shows first that the integrals of the equation $y''' + 2A(x)y' + [A'(x) + b(x)]y = 0$ for which $y(a) = 0$ are the solutions of a certain linear differential equation of second order; similarly for those integrals for which $y'(a) = 0$ or $y''(a) = 0$. For these classes of integrals certain results are proved, in particular the following theorem: If $A = A(x, \lambda) > 0$, $A' = \partial A(x, \lambda) / \partial x$, $b = b(x, \lambda) > 0$ are continuous in $-\infty < x < \infty$, $\Lambda_1 < \lambda < \Lambda_2$, and A is increasing in λ , $A(x, \lambda) \rightarrow \infty$ as $\lambda \rightarrow \Lambda_2$, and $a < b < c$, then the above equation has infinitely many solutions $y_n, y_{n+1}, \dots, y_{n+p}, \dots, y_{n+p}(a) = y_{n+p}(b) = y_{n+p}(c) = 0$, belonging to the parameter values $\lambda_n, \lambda_{n+1}, \dots, \lambda_{n+p}, \dots$ which converge to Λ_2 . The solution y_{n+p} has exactly $n+p$ zeros in (b, c) .

M. Golomb (Lafayette, Ind.).

Ivanenko, V. V. On a question of the general theory of ordinary differential equations in a complex region. *Kiiv. Derž. Ped. Inst. Nauk. Zap.* 16, Fiz.-Mat. Ser. no. 5 (1954), 13-20. (Ukrainian)

The author discusses the construction of the solution of the differential equation $y^{(m)} + p_1(x)y^{(m-1)} + \dots + p_m(x)y = 0$, in the following cases: (a) $p_i(x) = \varphi_i(x)/(x-a)^i$ ($i=1, \dots, m$), where $\varphi_i(x)$ is regular at $x=a$, in the form $y = \sum_{k=0}^{\infty} c_k(x-a)^{k+r}$; (b) $p_i(x) = \sum_{k=1}^{\infty} \beta_{ik}/x^k$, in the form $y = \sum_{k=0}^{\infty} c_k/x^{k+r}$ ($i=1, \dots, m$); and a few other cases of the regular singular points of the differential equation.

S. Kulik (Columbia, S.C.).

Demidovič, B. P. On some properties of the characteristic exponents of a system of ordinary linear differential equations with periodic coefficients. *Moskov. Gos. Univ. Uč. Zap.* 163, Mat. 6 (1952), 123-136. (Russian)

The system considered is (1) $\dot{x} = P(t)x$, where x is an n -vector and $P(t)$ a continuous matrix with period ω . The well known solutions rest upon the calculations of the characteristic exponents. The author gives first an explicit equation for these. He then relates them to certain averaging processes. More precisely, supposing $0 < \omega < \Omega$, and $\bar{P}(t)$ uniformly bounded on this interval, likewise

$\lim_{\omega \rightarrow 0} \omega^{-1} \int_0^\omega p_{ij}(t, \omega) dt = a_{ij}$ and $\|a_{ij}\| = A$, then one has Theorem 1: As $\omega \rightarrow 0$ the characteristic exponents of (1) tend to the characteristic roots of (2) $du/dt = Au$. He then proves Theorem 2: If $P(t)$ has period ω and either (a) $P(t)$ is symmetric with $P(t) = P(-t)$, or else (b) it is skew-symmetric with $P(t) = -P(t)$, then under (a) all solutions are bounded for all t , under (b) all solutions have period ω or 2ω .
S. Lefschetz (Mexico, D.F.).

Dobrotin, D. A. Some formulas for purely forced solutions of linear differential equations. Vestnik Leningrad. Univ. 10 (1955), no. 5, 45-48. (Russian)

Fedorov, G. F. Some new cases of solution of a system of two linear differential equations in finite form. Vestnik Leningrad. Univ. 1953, no. 11, 57-65. (Russian)

Suppose U_0, U_1, U_2 are 2×2 nilpotent matrix functions and $\varphi_0, \varphi_1, \varphi_2$ are scalar functions of x . Then the system $dY/dx = Y(\varphi_0 U_0 + \varphi_1 U_1 + \varphi_2 U_2)$ can be integrated by quadratures if

$$\varphi_0 = \frac{a^2}{\sigma_{01}} \varphi_1 + \left(\frac{a\sigma_{02} + \sigma_{012}}{\sigma_{01}} \right)^2 \frac{\varphi_2}{\sigma_{02}}$$

for some constant a . Here σ_{ij} , σ_{ijk} are the traces of $U_i U_j$, $U_i U_j U_k$. In proving this the author makes use of several identities involving the U_i , in particular of

$$U_2 = \frac{\sigma_{12}}{\sigma_{01}} U_0 + \frac{\sigma_{02}}{\sigma_{01}} U_1 + \frac{\sigma_{012}}{\sigma_{01}^2} (U_0 U_1 - U_1 U_0).$$

M. Golomb (Lafayette, Ind.).

* **Krein, M. G.** The basic propositions of the theory of λ -zones of stability of a canonical system of linear differential equations with periodic coefficients. Pam'yati Aleksandra Aleksandrovicha Andronova [In memory of Aleksandr Aleksandrovich Andronov], pp. 413-498. Izdat. Akad. Nauk SSSR, Moscow, 1955. 36.40 rubles.

Let $J = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$, where I_m is the unit matrix of m

columns and rows. Let $H(t)$ be a symmetric and periodic matrix of period T , $H(t+T) = H(t)$. The system (*) $dx/dt = \lambda J H(t)x$ is considered for $-\infty < t < \infty$. Let $\lambda = \lambda_0$ and suppose that $H_1(t)$ is also symmetric and periodic and close to $H(t)$ over $(0, T)$. Let all solutions of $dx/dt = \lambda_0 J H_1(t)x$ be bounded on $(-\infty, \infty)$ for any such $H_1(t)$. Then λ_0 is said to be a point of strong stability of (*). It is shown that these points form a set of open non-intersecting intervals which are called the λ -zones of stability. A very complete theory is given and also an account of the required theory of matrices and systems of ordinary differential equations. N. Levinson (Cambridge, Mass.).

Yoshizawa, Taro. On the stability of solutions of a system of differential equations. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. 29 (1955), 27-33.

Several notions of stability for the solution $x=0$ of a system $\dot{x} = f(x, t)$ with $f(0, t) = 0$ are defined; some simple examples and theorems are given, illustrating the relations between these notions. G. E. H. Reuter.

Maizel', A. D. On stability of solutions of systems of differential equations. Ural. Politehn. Inst. Trudy 51 (1954), 20-50. (Russian)

Compare the systems

- (1) $\dot{x} = P(t)x + L(x, t),$
- (2) $\dot{x} = P(t)x,$

where x, L are n -vectors, P is a bounded continuous matrix, and L a power series in the x , convergent in $t \geq t_0 > 0, \|x\| < H, H > 0$.

There are three stability criteria for the origin for (1) (actually equivalent) as follows: I. The system

$$\dot{x} = P(t)x + w(t)$$

for bounded w ($\|w\| < B$) has only bounded solutions (Perron). II. Existence of a positive definite Liapunov function for (2) whose time derivative is definite negative (Malkin). III. Let $x^j(t, t_0)$ be a fundamental system of solutions of (2) such that $\|x^j(t_0, t_0)\| = 1$; then

$$\|x^j(t, t_0)\| < K e^{-\alpha(t-t_0)},$$

where K, α are positive constants independent of t_0 (Persidskiĭ). These three criteria and their equivalence are extended by the author to conditional stability.

S. Lefschetz (Mexico, D.F.).

Vorovič, I. I. On stability of motion for constantly acting disturbances. Rostov. Gos. Univ. Uč. Zap. Fiz.-Mat. Fak. 18 (1953), no. 3, 99-105. (Russian)

The initial system dealt with is an n -vector system

$$(1) \quad \dot{x} = F(x; t; P(x; t))$$

where P is a k -vector representing a constantly acting disturbance. After Četaev [Stability of motion, Gostehizdat, Moscow, 1946] stability of the initial condition $x(0)$ is defined thus: Given ε and t_0 there exists a δ such that if $\|x(0) - \tilde{x}(0)\|$ and $\|P\| < \delta$ then $\|x(t) - \tilde{x}(t)\| < \varepsilon$ for $t > t_0$. The author transforms (1) to

$$(2) \quad \dot{\xi} = f(\xi; t; \eta),$$

where η is the disturbance vector. Assuming more or less that the components of f are power series in those of ξ, η he obtains highly complicated analytical conditions for the kind of stability under discussion. S. Lefschetz.

Gihman, I. I. Concerning a theorem of N. N. Bogolyubov. Ukrain. Mat. Ž. 4 (1952), 215-219. (Russian)

The author considers a vector system

$$(1) \quad \dot{x} = X(t, x, \lambda)$$

where $t \in [0, T]$, and x is in a certain region D of its space. He proves for such a system a theorem which includes an averaging theorem of Bogolyubov [On some statistical methods in mathematical physics, Akad. Nauk Ukrain. SSR, 1945; MR 8, 37] as well as a theorem on the continuous dependence of the solutions on λ [Petrovskii, Lectures on the theory of ordinary differential equations, 3rd ed., Gostehizdat, Moscow, 1949; MR 12, 334].

S. Lefschetz (Mexico, D.F.).

Moiseev, N. N. On a probability treatment of the concept "stability of motion". Rostov. Gos. Univ. Uč. Zap. Fiz.-Mat. Fak. 18 (1953), no. 3, 79-82. (Russian)

In the n -vector system $\dot{x} = X(t; x)$, $X(t; 0) = 0$ let the origin be the motion whose stability is to be defined. This is done in terms of a system $\dot{x} = X + F(t)$. Let the initial random point x^0 have a mathematical expectation 0 with dispersion $\sigma^0 = \|x^0\|^2$. As for F , let it have a correlation function $k_{F,F}(t_1, t_2)$. Stability is then defined as follows: given ε there exist δ, η such that $\sigma^0 < \delta, \|k_{F,F}\| < \eta$ imply that $\sigma(t) < \varepsilon$ for $t > t_0$ ($\sigma(t)$ is the dispersion of $x(t)$). S. Lefschetz (Mexico, D.F.).

Sokolov, V. M. On periodic oscillations of Lyapunov systems in a special case. Ural. Politehn. Inst. Trudy 51 (1954), 12-19. (Russian)
Study of the periodic solutions of

$$(1) \quad \dot{x} = Ax + X(x),$$

where x, X are $(n+4)$ -vectors, A is a constant matrix with two pairs of complex characteristic roots of the form $\pm \lambda i, \pm k \lambda i$, k an integer, and the components of X are power series with terms of degree ≥ 2 . Moreover, it is assumed that there is a first integral $H(x) = \text{const}$, where H is a power series which actually begins with terms of the second degree. A first change of variable reduces (1) to the form

$$\begin{aligned} \dot{x}_1 &= -k\lambda y_1 + \dots, & y_1 &= k\lambda x_1 + \dots, \\ \dot{x}_2 &= -\lambda y_2 + \dots, & y_2 &= \lambda x_2 + \dots, \\ \dot{z} &= Bz + \dots, \end{aligned}$$

where \dots are analytic of the same type as the components of X , z is an n -vector and B is a constant matrix without characteristic roots of the form $\pm p \lambda i$, p an integer. The first integral may then be taken as

$$k_1(x_1^2 + y_1^2) + x_2^2 + y_2^2 + W(x, y, z) = \mu^2,$$

where k_1 is a positive constant and W is a power series beginning with terms of the third degree in x_i, y_j and of the second degree in the components of z . Then the change of variables

$$x_1 = r x, y_1 = r y, x_2 = r \cos \theta, y_2 = r \sin \theta, z = r u,$$

reduces the system to the form

$$\begin{aligned} \frac{dx}{d\theta} &= -ky + \mu^{m-1}P_m + \mu^m P_{m+1} + \dots, \\ \frac{dy}{d\theta} &= kx + \mu^{m-1}Q_m + \dots, \\ \frac{du}{d\theta} &= Cu + \mu^{m-1}R_m + \dots, \\ \omega &= l \text{ for } l < m, = m \text{ for } l \geq m, \end{aligned}$$

where P, \dots are analytic in t, x, y, u , and m is the least degree term in the first four \dots , and l the same in the fifth. At this stage the system is of a type studied for periodic solutions by Malkin [The methods of Lyapunov and Poincaré in the theory of non-linear oscillations, OGIZ, Moscow-Leningrad, 1949; MR 12, 28].

S. Lefschetz (Mexico, D.F.).

Loud, W. S. On periodic solutions of Duffing's equation with damping. J. Math. Phys. 34 (1955), 173-178.

Consider (E): $\ddot{x} + c\dot{x} + x + \beta x^3 = p(t)$, with $c > 0, \beta > 0, p(t)$ periodic with zero average, $\max |p(t)| = 1$. By a theorem of M. L. Cartwright and J. E. Littlewood [Ann. of Math. (2) 48 (1947), 472-494; MR 9, 35], (E) has a unique periodic solution to which all solutions converge when $t \rightarrow +\infty$, provided that $c > c_0(\beta)$. The author gives an estimate for $c_0(\beta)$ by showing that "convergence" takes place if $c^2 > 48A\beta$, where

$$A = \min(1 + 4c^{-1}, 1 + 4c^{-2}, \beta^{-4} + 4c^{-2}).$$

G. E. H. Reuter (Manchester).

Reissig, Rolf. Über eine nichtlineare Differentialgleichung 2. Ordnung. II. Math. Nachr. 14 (1955), 65-71.

The author again re-examines one of his previous

proofs and extends slightly a previous result [Math. Nachr. 13 (1955), 313-318; MR 17, 38]. The differential equation is $\ddot{u} + F(u) + u = E(t)$, which is equivalent to the differential equation studied in the paper referred to above. E is a continuous, periodic function. F is a continuous, strictly increasing function with $F(0) = 0$. It is shown: if

$$\lim_{v \rightarrow +\infty} \{F(v) - F(-v)\} > \max E(t) - \min E(t),$$

then the equation above has a unique periodic solution with period that of E and all solutions are asymptotic to this periodic solution as $t \rightarrow +\infty$. J. P. LaSalle.

Nikitin, A. K. Nonlinear oscillations of a system with a disturbing force consisting of two harmonics. Rostov. Gos. Univ. Uč. Zap. Fiz.-Mat. Fak. 18 (1953), no. 3, 55-63. (Russian)

The equation $m\ddot{x} + f(x) = e(t)$ where $e(t)$ consists of two harmonics is solved approximately. The author assumes x to be a linear combination with varying coefficients of harmonics with the same frequencies as $e(t)$ and obtains a system for the coefficients in which he replaces all periodic terms by their mean values over a period. The stability of these periodic solutions is also discussed. As an example, the case $f(x) = \alpha x + \beta x^3$ is worked out.

H. A. Antosiewicz (Washington, D.C.).

★ **Alfzerman, M. A.; and Smirnova, I. M.** On application of small-parameter methods for investigation of periodic regimes in systems of automatic control not having a small parameter. Pamyati Aleksandra Aleksandroviča Andronova [In memory of Aleksandr Aleksandrovič Andronov], pp. 77-92. Izdat. Akad. Nauk SSSR, Moscow, 1955. 36.40 rubles.

★ **Bautin, N. N.** Dynamic models of free clock movements. Pamyati Aleksandra Aleksandroviča Andronova [In memory of Aleksandr Aleksandrovič Andronov], pp. 109-172. Izdat. Akad. Nauk SSSR, Moscow, 1955. 36.40 rubles.

★ **Železcov, N. A.** On the theory of the kipp-relay. Pamyati Aleksandra Aleksandroviča Andronova [In memory of Aleksandr Aleksandrovič Andronov], pp. 215-229. Izdat. Akad. Nauk SSSR, Moscow, 1955. 36.40 rubles.

★ **Neimark, Yu. I.** On periodic motions of relay systems. Pamyati Aleksandra Aleksandroviča Andronova [In memory of Aleksandr Aleksandrovič Andronov], pp. 242-273. Izdat. Akad. Nauk SSSR, Moscow, 1955. 36.40 rubles.

★ **Tal', A. A.** Influence of auto-regulation and of action upon the derivative in processes of direct control. Pamyati Aleksandra Aleksandroviča Andronova [In memory of Aleksandr Aleksandrovič Andronov], pp. 282-299. Izdat. Akad. Nauk SSSR, Moscow, 1955. 36.40 rubles.

★ **Cypkin, Ya. Z.** A frequency method of investigating periodic regimes of relay systems of automatic control. Pamyati Aleksandra Aleksandroviča Andronova [In memory of Aleksandr Aleksandrovič Andronov], pp. 383-410. Izdat. Akad. Nauk SSSR, Moscow, 1955. 36.40 rubles.

Order ★ **Frequency response.** Edited by Rufus Oldenburger. The Macmillan Company, New York, 1956. xii+372 pp. (1 plate). \$7.50.

A collection of 28 papers on frequency response, control systems, etc. Some of them have already appeared in Trans. A.S.M.E. 76 (1954), no. 8.

Ascoli, Guido. Sopra un principio di trasformazione integrale dei problemi differenziali ed alcune sue applicazioni. Ann. Mat. Pura Appl. (4) 40 (1955), 167-182.

If $x(t)$ satisfies an ordinary differential equation, $L_t x + \lambda M_t x = 0$, where L_t and M_t are differential operators and λ is a constant, and $K(t, u)$ satisfies a partial differential equation of the form $(L_t Q_u - M_t P_u)K = 0$, where P_u and Q_u are differential operators, then

$$y(u) = \int_a^b M_t K(t, u) x(t) dt$$

satisfies the ordinary differential equation $P_u y + \lambda Q_u y = 0$ provided that L_t is self-adjoint. This is applied to obtain integral equations and integral relations (mostly known) for Mathieu functions, Hermite and Legendre polynomials, Bessel functions. A. Erdélyi (Pasadena, Calif.).

Rasulov, M. L. Expansion of an integrable function in the fundamental functions of a boundary problem of an ordinary differential equation. Izv. Akad. Nauk Azerbaidžan. SSR. 1953, no. 6, 3-28. (Russian. Azerbaijani summary)

The boundary-value problem considered concerns the n th order differential equation

$$(*) \quad \sum_{i=0}^{n-1} P_i(x, \lambda) y^{(i)}(x, \lambda) = f(x), \\ P_0(x, \lambda) = 1, \quad a_1 \leq x < a_{m+1},$$

with coefficients continuous except for finite jumps at the m points a_i , $i=2, \dots, m+1$, $a_1 < a_2 < \dots < a_{m+1}$. The boundary conditions imposed are

$$\sum_{k=1}^n \sum_{j=1}^n \{ \alpha_{ij}^{(k)}(\lambda) y_k^{(j-1)}(a_k, \lambda) + \beta_{ij}^{(k)}(\lambda) y_k^{(j-1)}(a_{k+1}, \lambda) \} = 0,$$

$i=1, \dots, mn$, where y_k denotes a solution of (*) on (a_k, a_{k+1}) . The investigation consists of a generalization of the classical treatment of the problem which is the restriction of the one above to a single interval (a_k, a_{k+1}) . Analogous results are obtained. A Green's function is determined and is shown to be a meromorphic function of λ provided the coefficients of (*) and the α 's and β 's are analytic in λ with the rank of the mn by $2mn$ matrix of the α 's and β 's being mn . A necessary and sufficient condition for the existence of a solution to the problem at $\lambda = \lambda_0$ is given. Assuming that (i) $P_i(x, \lambda) = P_i^{(0)}(x, \lambda)$ on (a_j, a_{j+1}) , $P_i^{(0)}(x, \lambda) = \lambda^i \sum_{k=0}^{j-1} p_{i,k}^{(0)}(x) \lambda^{-k}$ on (a_j, a_{j+1}) where the $p_{i,k}^{(0)}(x)$ are suitably differentiable, (ii) the roots of the characteristic equation of (*) are distinct on each interval, and (iii) the rank of the matrix of α 's and β 's is mn while the α 's and β 's themselves are polynomials in λ of degree no greater than n , the author proves theorems concerning the rate of growth of the eigenvalues, the asymptotic expansion of the Green's function, and the expansion of an integrable function in terms of the Green's function and certain of its partial derivatives.

N. D. Kazarinoff (Lafayette, Ind.).

Kay, I.; and Moses, H. E. The determination of the scattering potential from the spectral measure function.

II. Point eigenvalues and proper eigenfunctions. Nuovo Cimento (10) 3 (1956), 66-84.

Reviewed earlier as a report [MR 17, 155].

Marčenko, V. A. On reconstruction of the potential energy from phases of the scattered waves. Dokl. Akad. Nauk SSSR (N.S.) 104 (1955), 695-698. (Russian)

Let the potential $V(x)$ be real and let $\int_0^\infty x |V(x)| dx < \infty$. Let $u'' + (\mu - V(x))u = 0$, $u(0) = 0$, have discrete eigenvalues at μ_k . Let $\lambda = \mu^2$, let $\lambda_k = (-\mu_k)^2$ and let the k th eigenfunction be denoted by $u(\lambda_k, x)$. As $x \rightarrow \infty$ let

$$u(\lambda, x) \sim (2/\pi)^{1/2} \sin(\lambda x + \eta_0(\lambda)) \text{ for } \lambda > 0$$

and let $u(\lambda_k, x) \sim m_k \exp(-\lambda_k x)$. Suppose $\eta_0(\lambda)$, λ_k and m_k are given. Then $V(x)$ is determined from $2A(x, x) = \int_x^\infty V(t) dt$, where $A(x, y)$ is the solution of

$$f(x, y) + A(x, y) + \int_x^\infty A(x, t) f(t, y) dt = 0$$

and

$$f(x, y) = f(x+y) = \sum_k m_k^2 e^{-\lambda_k(x+y)} - \frac{1}{2\pi} \int_{-\infty}^\infty [e^{2i\eta_0(\lambda)} - 1] e^{i\lambda(x+y)} d\lambda.$$

For $\lambda < 0$, $\exp(2i\eta_0(\lambda)) = \exp(-2i\eta_0(-\lambda))$.

N. Levinson (Cambridge, Mass.).

★ **Krein, M. G.** On some cases of the effective determination of the density of a nonuniform string by its spectral function. Translated by Morris D. Friedman, 2 Pine St., West Concord, Mass., 1955. 7 pp. \$3.00. Translation of Dokl. Akad. Nauk SSSR (N.S.) 93 (1953), 617-620; MR 15, 796.

See also: Bahtin and Krasnosel'skiĭ, p. 803; Humblet, p. 811.

Partial Differential Equations

Haimovici, Mendel. Quelques propriétés des éléments intégraux d'un système de Pfaff du II^e genre. Acad. Repub. Pop. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. 7 (1955), 301-311. (Romanian. Russian and French summaries)

Haimovici, M. Quelques propriétés du prolongement d'un système de Pfaff du II^e genre. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. 7 (1955), 583-594. (Romanian. Russian and French summaries).

If $\theta_1 = \dots = \theta_n = 0$ is a linear Pfaffian system and θ_i' are the derived forms of the left members, then θ_i' are congruent modulo the system to forms $\varphi_i = A_{ij} \omega_j$, where $\theta_1, \dots, \theta_n, \omega_1, \dots, \omega_{n-s}$ are a set of n linearly independent linear forms, n being the number of independent variables. The present two papers consider the case where $\varphi_i = \varphi_{i1} \omega_1 + \varphi_{i2} \omega_2$, the φ 's and ω 's being linear. Numerous formal results about so-called reducing Pfaffian systems are given. Generalizations and applications to the study of the solution of the original Pfaffian system are promised.

J. M. Thomas (Durham, N.C.).

Ważewski, T. Sur certaines inégalités aux dérivées partielles relatives aux fonctions possédant la différentielle approximative. Ann. Polon. Math. 2 (1955), 219-233 (1956).

A function $f(x, y)$ is said to have an ε -approximate partial derivative α with respect to x at (x_0, y_0) if

$$\limsup |f(x, y) - f(x_0, y) - \alpha(x - x_0)| / |x - x_0| \leq \varepsilon$$

as $x \rightarrow x_0$. Similarly ε -approximate differentials and gradients are defined. One of the main theorems gives the "integration" of a system of ε -approximate differential inequalities (involving the maximal solution of a corresponding system of ordinary differential equations). This theorem is used to prove, for a system of partial differential equations $\partial z_k / \partial x_j =$

$$G_{kj}(x_1, \dots, x_l; y_1, \dots, y_m; z_1, \dots, z_n; \partial z_k / \partial y_1, \dots, \partial z_k / \partial y_m),$$

where G_{kj} does not depend on $\partial z_s / \partial y_r$ for $s \neq k$, an approximation theorem concerning the limits of ε -approximate solutions as $\varepsilon \rightarrow 0$. The last section concerns a method for obtaining ε -approximate solutions of an equation $\phi = f(x, y, z, q)$.
P. Hartman (Baltimore, Md.).

Fava, Franco. Varietà integrali di particolari sistemi di due equazioni di Laplace per una funzione di tre variabili. Univ. e Politec. Torino. Rend. Sem. Mat. 14 (1954-55), 189-237.

By a suitable transformation, the system

$$\begin{aligned} x_{ww} &= a_1 x_u + a_2 x_v + a_3 x_w + ax \\ x_{uv} &= L x_{uw} + b_1 x_u + b_2 x_v + b_3 x_w + bx \end{aligned}$$

is reduced to a system which is (*) with $L=0$. From the conditions of integrability on the reduced system, a system of eight partial differential equations is formed which involve only the coefficients of the reduced system. With the coefficients so defined, the integral varieties and characteristic surfaces of the reduced system are discussed. In the second equation of equations [1.2] and [4.2], the term $a_3 x_{vw}$ is missing.
R. T. Herbst.

Fava, Franco. Sulle varietà integrali del sistema

$$\begin{aligned} \{x_{ww} &= a_1 x_u + a_2 x_v + a_3 x_w + ax \\ x_{uv} &= L x_{uw} + b_1 x_u + b_2 x_v + b_3 x_w + bx. \end{aligned}$$

Univ. e Politec. Torino. Rend. Sem. Mat. 14 (1954-55), 239-256.

By means of the operator $(\)_o = (\)_v - L(\)_w$, the system displayed in the title is shown to be formally equivalent to the same system with $L=0$. A discussion of integral varieties and characteristic surfaces follows that given in the paper reviewed above.
R. T. Herbst.

Danilyuk, I. I. On some questions of the theory of elliptic systems of differential equations of first order on surfaces. Dokl. Akad. Nauk SSSR (N.S.) 105 (1955), 11-13. (Russian)

In this note the author sketches, without proofs, a theory of elliptic systems of first-order partial differential equations on a (orientable and twice Hölder continuously differentiable) surface R . The system is written in the form

$$\frac{\partial u^i}{\partial x^j} = \sum_{k=1}^2 \alpha_k^j \frac{\partial u^k}{\partial x^i} \quad (i=1, 2),$$

where α_k^j is a tensor. (The components α_k^j are assumed to be Hölder continuously differentiable.) It is called elliptic if the following condition is satisfied: set $\varepsilon^{11} = \varepsilon^{22} = 0$, $\varepsilon^{12} = -\varepsilon^{21} = 1$ and $\bar{a}^{ij} = \sum \varepsilon^{ij} \alpha_k^j$; then the matrix (\bar{a}^{ij}) should be positive definite. The author states some (essentially known) properties of solutions of (1), like

interiority, maximum principle, and Harnack's convergence theorem. The following result is used in order to construct solutions with given singularities and also analogues of Abelian differentials on closed surfaces. Theorem 3: Let $W(\zeta) = U^2 + iV^2$ be a function defined in a plane domain which is continuously differentiable and has a non-vanishing Jacobian except at isolated points. Then there exists a solution of (1) which is of the form $u^1 + iu^2 = W[\chi(\zeta)]$, where $\zeta = \chi(z)$ is a homeomorphism. [In the most important case, $W(\zeta)$ analytic, the theorem has been proved previously by Boyarskii, same Dokl. 102 (1955), 871-874; MR 17, 157.]

In the space of solutions of (1) defined on a plane domain or on a surface the author introduces a scalar product which makes this space into a complete Hilbert space. He is then able to apply the methods of Bergman [Math. Ann. 86 (1922), 238-271], and Bergman and Schiffer [Duke Math. J. 14 (1947), 609-638; MR 9, 187] and uses the kernel function to obtain an analogue of the Cauchy integral formula and to solve the oblique derivative problem.

Finally, he states two mapping theorems. A simply connected closed surface can be mapped by a solution of (1) onto the Riemann sphere. A surface R of genus zero can be mapped by a solution of (1) onto a plane domain bounded by parallel slits or onto a limit of such domains.

No conformal structure is assumed on R . The reviewer remarks, however, that one can introduce on R a conformal structure determined by equation (1). On the so constructed Riemann surface solutions of (1) become pseudoanalytic functions and the results of the author would remain true under weaker differentiability assumptions.
L. Bers (New York, N.Y.).

Beckert, Herbert. Systeme partieller linearer elliptischer Differentialgleichungen erster und höherer Ordnung mit zwei unabhängigen Variablen. Math. Nachr. 12 (1954), 257-272.

This paper is a continuation of a preceding one [Math. Nachr. 5 (1951), 173-208; MR 13, 748] on general boundary problems for elliptic systems of linear, first-order partial differential equations in two independent variables. In that paper, the dependent variables were transformed in ways dictated by the boundary conditions, and the equations then were recombined, to put the differential equations into a normal form from which integral equations could be easily set up. One object of the present paper is to clarify further the connections, which are rather involved, between these integral equations and the original problem, particularly in the case of multiply connected domains. Secondly, the author's theory is applied to proving, under few restrictions as to differentiability, the fact that the solutions of elliptic systems of equations with analytic coefficients are analytic. Finally, a procedure is described to reduce boundary problems for elliptic systems of any order to corresponding boundary problems for first-order elliptic systems and thus, indeed, to integral equations of the sort treated.
A. Douglis (New York, N.Y.).

Weinstein, Alexander. On a class of partial differential equations of even order. Ann. Mat. Pura Appl. (4) 39 (1955), 245-254.

It is one of the purposes of this paper to show that it is possible to find a decomposition of a biharmonic function into a sum of two functions each satisfying an equation of the second order which occurs in generalized axially

symmetric potential theory. The author uses for both of the differential operators

$$\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_m^2} + \frac{\partial^2}{\partial y^2} + \frac{h}{y} \frac{\partial}{\partial y} \quad (\text{elliptic operator}),$$

$$-\infty < h < \infty,$$

$$\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_m^2} - \frac{\partial^2}{\partial y^2} + \frac{h}{y} \frac{\partial}{\partial y} \quad (\text{hyperbolic operator}),$$

the same notation L_k because only common properties of both operators are used in the major part of the paper. A solution of the partial differential equation $L_k u = 0$ is denoted as u^k or $u^{(k)}$. Then the following fundamental recursion formulas hold:

$$(*) \quad u_y^k(x, y) = y u^{k+2}(x, y), \quad (**) \quad u^k(x, y) = y^{1-k} u^{k-2}(x, y).$$

The recursion formula (*) defines $u^{(k+2)}$ in terms of $u^{(k)}$. The relation

$$u^k(x, y) = \int_b^y \eta u^{k+2}(x, \eta) d\eta + f(x, b),$$

where $f(x, b)$ are the values which $u^{(k)}$ takes for $y=b$, may be interpreted as an integral equation which yields a function $u^{(k)}$ from a function $u^{(k+2)}$. The author proves the theorem: the general solution of the equation

$$L_\alpha L_\beta w = 0, \quad \beta \neq \alpha - 2,$$

is given by the formula $w = u^{(\beta)} + u^{(\alpha-2)}$. Here $u^{\alpha-2} u^\beta$ can be replaced by $u^{\alpha-2} + h$ and $u^\beta - h$ respectively, where h is an arbitrary harmonic function of the variables x_1, x_2, \dots, x_m . The decomposition $u^{(\beta)} + u^{(\alpha-2)}$ is valid in any cylindrical domain of $(m+1)$ -dimensional space with its base in the subspace $y=b=\text{constant}$ and with generators parallel to the y -axis, this cylinder lying entirely in the domain of regularity of w . The result can be generalized. Considering the differential equation

$$L_\alpha L_\alpha \cdots L_\alpha \omega = 0, \quad \alpha_r \neq \alpha_j - 2(r-j), \quad j < r = 2, 3, \dots, n,$$

the author gets the general solution

$$(***) \quad w = u^{\alpha_n} + u^{\alpha_n-2} + u^{\alpha_n-4} + \cdots + u^{\alpha_n-2(n-1)}.$$

Each of the functions u on the right hand side of (***) is determined up to a harmonic function $h(x_1, x_2, \dots, x_m)$ such that their sum $\sum h$ is identically zero. The proof is obtained from the case $n=2$ by induction. As special cases the polyharmonic equation, the iterated wave equation, the iterated Euler-Poisson-Darboux equation and the iterated generalized axially symmetric potential equation are discussed. *M. Pinl (Cologne).*

Hartman, Philip; and Wintner, Aurel. Binary, linear, elliptic, partial differential equations. *Duke Math. J.* 22 (1955), 515-524.

In the first part the authors are concerned with reducing the smoothness hypotheses under which an elliptic linear equation

$$(1) \quad (A\phi_x + B_1\phi_y)_x + (B_2\phi_x + C\phi_y)_y + E\phi = 0$$

can be transformed, by introducing new independent variables, into the form

$$(2) \quad (\phi_u/g + \gamma\phi_v)_u + (-\gamma\phi_u + \phi_v/g)_v + \tilde{E}\phi = 0,$$

and the self-adjoint equation

$$(3) \quad (\phi_u/g)_u + (\phi_v/g)_v + F\phi = 0 \quad (g > 0)$$

into the normal form

$$(4) \quad \psi_{uu} + \psi_{vv} + H\psi = 0,$$

by introducing a new unknown function. All equations are considered as abbreviations for their "integrated forms". Thus a C^1 function ψ satisfies (4) if

$$\int_J \psi_u dv - \psi_v du = \iint_T H \psi du dv$$

for every domain T bounded by a piecewise smooth curve J . Results: (1) can be transformed into (2) if the metric $A dy^2 - (B_1 + B_2) dx dy + C dx^2$ has a continuous curvature in the sense of Weyl; (3) can be transformed into (4) if g is of class C^1 and satisfies the integrated form of the Poisson equation $g_{uu} + g_{vv} = f$ for some continuous f .

In the second part it is shown that for $g > 0$ and of class C^1 equation (4) need not have non-constant C^2 solutions. Similarly, the system $g\psi_u = \phi_u$, $g\psi_v = -\phi_v$ need not have a non-constant C^1 solution.

L. Bers (New York, N.Y.).

Golomb, Michael. A note on linear vector spaces of mappings with positive Jacobians. *Proc. Amer. Math. Soc.* 5 (1954), 536-538.

Let F be a real linear vector space of continuously differentiable mappings $f = u + iv$ defined in a plane domain D . Assume that the Jacobian of each mapping in F is non-negative and vanishes only at points at which the Jacobian matrix has rank zero. The author proves that if F contains two mappings $f_1 = u_1 + iv_1$ and $f_2 = u_2 + iv_2$ such that $\partial(v_1, v_2)/\partial(x, y) \neq 0$, then all mappings of the family F satisfy a unique elliptic system of the form

$$u_x - a_{11}v_x - a_{12}v_y = 0, \quad u_y + a_{21}v_x + a_{22}v_y = 0$$

where the coefficients a_{ik} are continuous. *L. Bers.*

Satunov, M. P. On the resolvent of an elliptic operator. *Mat. Sb. N.S.* 37(79) (1955), 459-470. (Russian)

Let L_x and L_y be the differential operators

$$p^{-1}(x) D_x \phi(x) D_x + q^{-1}(y) D_y \phi(y) D_y \quad (p > 0, q > 0),$$

defined in bounded intervals and let R_x^λ and R_y^λ be the resolvents of $L_x + k r^2(x)$ and $L_y + k \varrho^2(y)$ ($\text{Im } k > 0$; $r, \varrho \neq 0$ and real) associated with real non-negative Sturm-Liouville boundary conditions. It is shown that $R_k = -(2\pi i)^{-1} \int_{-\infty}^{+\infty} R_x^\lambda R_y^\lambda (r^2(x) + \varrho^2(y)) d\lambda$ is the resolvent of the differential operator $(r^2(x) + \varrho^2(y))^{-1} (L_x + L_y)$, associated with the product of the boundary conditions (in a rectangle) and that

$$\text{Im } R_k = -\pi^{-1} \int_{-\infty}^{+\infty} \text{Im } R_x^\lambda \text{Im } R_y^\lambda (r^2(x) + \varrho^2(y)) d\lambda.$$

L. Gårding (Lund).

Nirenberg, Louis. Remarks on strongly elliptic partial differential equations. *Comm. Pure Appl. Math.* 8 (1955), 649-675.

Notations: sur un ouvert \mathcal{D} de R^n , H_m est l'espace des fonctions $u \in L^2$ ainsi que leurs dérivées (distribution) d'ordre $\leq m$; $\|u\|_m$ est la norme hilbertienne naturelle; \bar{H}_m est l'adhérence dans H_m des fonctions à support compact dans \mathcal{D} ; pour $u, v \in H_m$, on considère la forme sesquilinéaire

$$B(u, v) = \sum_{\alpha, \beta=0}^m (D^\alpha u, a D^\beta v), \quad a^{\alpha\beta} \in L^\infty$$

($D^\lambda u$ est une dérivation d'ordre λ); la forme $B(u, v)$ définit un opérateur différentiel L ; on suppose L elliptique au sens

$$\operatorname{Re}(-1)^m \sum \xi^m a^{mm} \xi^m \geq c_0 |\xi|^{2m}, \quad c_0 > 0.$$

Le problème de Dirichlet "faible" est de trouver u dans \dot{H}^m solution de $Lu=f$, f donné dans L^2 [cf. Gårding, Math. Scand. 1 (1953), 55-72; MR 16, 366] (Rev.: plus généralement $f \in (\dot{H}^m)'$ dual de \dot{H}^m); le problème de Neumann faible est de trouver $u \in H_m$ solution de $B(u, v) = (f, v)$ pour tout $v \in H_m$; pour des problèmes plus généraux cf. Aronszajn, Browder ou le Rev.

L'A. étudie dans cet article la régularité de u , (i) dans \mathcal{D} (ii) à la frontière de \mathcal{D} , moyennant des hypothèses de régularité convenable sur les coefficients, la frontière et f . Résultat (i): soit u localement L^2 solution de $Lu=f$, f étant localement dans H_p (resp. $(\dot{H}_p)' = H_{-p}$), alors u est localement dans H_{p+2m} (resp. H_{-p+2m}). Résultat (ii): si u est solution du problème de Dirichlet ou de Neumann "faible" et si f est dans H_p alors $u \in H_{p+2m}$. (L'A. montre un peu moins; un lemme simple donne le résultat actuel.) Conséquence de (ii): en prenant p assez grand, on a une solution u autant de fois différentiable que l'on veut dans $\bar{\mathcal{D}}$ donc solution usuelle des problèmes aux limites [autre conséquence: on retrouve les inégalités de O. V. Guseva, Dokl. Akad. Nauk SSSR (N.S.) 102 (1955), 1069-1072; MR 17, 161].

Méthode de démonstration: on considère u^h définie par $u^h(x) = h^{-1}(u(x+h) - u(x))$, $h = (0, \dots, 0, h, 0, \dots, 0)$; des calculs simples et l'ellipticité de $B(u, v)$ (intervenant sous la forme de l'inégalité de Gårding) montrent que u^h demeure localement dans un ensemble borné de H_m , d'où $D^k u \in H_m$; l'introduction de u^h remplace les "mollifiers" de Friedrichs; son avantage est de pouvoir s'appliquer au cas des problèmes aux limites: on se ramène par carte locale au cas d'un demi-espace $x_n > 0$, et on translate u dans des directions parallèles à $x_n = 0$. On montre ainsi que $\partial u / \partial x_i$, $i < n$, est localement dans H_m au voisinage de la frontière; par itération du procédé, on montre que toutes les dérivées tangentielles d'ordre $\leq r+m$ sont dans H_m ; la seule considération de $L = \sum a_p(0) D^p$ permet de passer aux autres dérivées, d'où le résultat.

Généralisation: la théorie s'applique aux formes continues et elliptiques sur l'espace produit $\dot{H}_{s_1} \times \dots \times \dot{H}_{s_n}$. J. L. Lions (Nancy).

Bers, Lipman. Local behavior of solutions of general linear elliptic equations. Comm. Pure Appl. Math. 8 (1955), 473-496.

Soit $L = \sum a_p(x) D^p$, $|p| \leq m$, un opérateur elliptique (au sens: $\sum_{|p|=m} a_p(x) \xi^p > 0$ pour $\xi \neq 0$) défini au voisinage de 0 dans R^n ; les a_p sont Hölder continus au voisinage de 0; les a_p , $|p| = m$, vérifient une condition de Hölder uniforme, d'exposant ϵ , $0 < \epsilon < 1$. Une fonction φ est dite s'annuler en 0 à un ordre fini s'il existe un entier $N \geq 1$ tel que

$$(a) \quad \varphi(x) = O(|x|^N), \quad \lim_{x \rightarrow 0} |x|^{-N-1} \varphi(x) = \infty, \quad x \rightarrow 0.$$

Théorème 1: soit φ solution usuelle (m fois continûment différentiable) de $L\varphi=0$, s'annulant en 0 à un ordre fini. Il existe alors un polynôme homogène p_N de degré N , non nul, solution de $L_0 p_N = 0$ ($L_0 = \sum a_p(0) D^p$), tel que

$$\varphi - p_N = O_m(|x|^{N+\epsilon})$$

($f = O_m(|x|^A)$ signifie: $D^q f(x) = O(|x|^{A-|q|})$, $0 \leq |q| \leq A$, $x \rightarrow 0$). (Dans le cas des coefficients analytiques, ce théorème résulte de l'analyticité de φ , démontrée par John.)

Principe de la démonstration: on se ramène d'abord au cas $n > m$ par une méthode de descente. Soit alors N avec (a). On a $L_0 \varphi = O(|x|^{N-m+\epsilon})$. Soit J solution élémentaire de

L_0 , analytique pour $x \neq 0$, avec $J(tx) = t^{m-n} J(x)$, $t > 0$, si $n > m$ ou si $n \leq m$ et n impair, et $J(x) = p(x) \log |x| + \Omega(x)$, $p =$ polynôme homogène de degré $m-n$, $\Omega(tx) = t^{m-n} \Omega(x)$, $t > 0$, si $n \leq m$ et n pair. En utilisant J et des évaluations de produits de composition ("potential theoretical Lemmas") on en tire

$$\varphi = p_N + O_{m-1}(|x|^{N+\epsilon}),$$

p_N polynôme homogène de degré N , $L_0 p_N = 0$. On montre que p_N n'est pas nul. Soit $\psi = \varphi - p_N$. On a $\psi = O(|x|^{N+\epsilon})$, $L\psi = -Lp_N = (L_0 - L)p_N$. Grâce au fait que $L_0 - L$ a ses coefficients nuls à l'origine et en utilisant des majorations a priori dues à Douglas et Nirenberg (cf. résumé ci après), on en déduit que les dérivées d'ordre m de ψ sont $O(|x|^{N-m+\epsilon})$, d'où le théorème.

Résultats complémentaires avec des hypothèses plus fortes sur les coefficients de L . L'A. étudie ensuite les singularités des solutions de $L\varphi=0$; sous les hypothèses du Théorème 1 soit φ solution de $L\varphi=0$ définie dans $0 < |x| \leq R$:

Premier cas:

$$\begin{aligned} \varphi &= o(|x|^{m-n}) \text{ si } n > m \text{ ou } n \text{ impair,} \\ \varphi &= o(|x|^{m-n} \log |x|) \text{ si } n \leq m \text{ et } n \text{ pair.} \end{aligned}$$

Alors l'origine est singularité apparente de φ .

Deuxième cas:

$$\varphi = O(|x|^{m-n-1+\epsilon}), \quad 1-\epsilon < \lambda < 1.$$

Alors

$$\varphi = cJ + O_m(|x|^{m-n-1+\epsilon}), \quad c = \text{constante.}$$

J. L. Lions (Nancy).

Douglas, Avron; and Nirenberg, Louis. Interior estimates for elliptic systems of partial differential equations. Comm. Pure Appl. Math. 8 (1955), 503-538.

On considère dans un ouvert \mathcal{D} de R^n un système différentiel $U = (u_1, \dots, u_N) \rightarrow (L_i U)$, $i = 1, \dots, N$, linéaire. Les A. introduisent la très générale notion d'ellipticité suivante: le système est dit elliptique s'il existe des entiers s_i , t_j tels que $L_i U = \sum l_{ij} u_j$, ordre $(l_{ij}) = s_i + t_j$ ($l_{ij} = 0$ si $s_i + t_j < 0$) et $\det(l_{ij}'(x, \xi)) \neq 0$ pour $\xi = (\xi_1, \dots, \xi_n) \neq 0$, pour tout $x \in \mathcal{D}$, $l_{ij}'(x, \xi)$ étant le polynôme correspondant à la partie principale de l_{ij} . La définition usuelle correspond au cas où tous les s_i sont égaux.

Notations: d_P = distance de $P \in \mathcal{D}$ à la frontière de \mathcal{D} ; si $P, Q \in \mathcal{D}$, $d_{PQ} = \min(d_P, d_Q)$. Pour u ayant des dérivées continues d'ordre i dans \mathcal{D} , on pose: $M_{p,i}(u) = \sup_{P \in \mathcal{D}} d_P^{p+i} |D^i u(P)|$, $|i| = i$, $P \in \mathcal{D}$, p entier, $p+i \geq 0$. Si u a ses dérivées d'ordre i Hölder continues (exposant α) sur tout compact de \mathcal{D} , on pose

$$M_{p,i+\alpha}(u) = \sup_{|i|=i; P, Q \in \mathcal{D}} d_{PQ}^{p+i+\alpha} |P-Q|^{-\alpha} |D^i u(P) - D^i u(Q)|.$$

Enfin

$$|u|_{p,i} = \sum_{j=\max(0, -p)}^i (j!)^{-1} M_{p,j}, \quad |u|_{p,i+\alpha} = |u|_{p,i} + M_{p,i+\alpha}.$$

Pour $p \geq 0$, les fonctions $[a]$ fois continûment différentiables dans \mathcal{D} , vérifiant des conditions de Hölder d'exposant $\alpha - [a]$ sur tout compact de \mathcal{D} et pour lesquelles $|u|_{p,a} < \infty$ forment un espace de Banach noté $C_{p,a}$.

Théorème essentiel: L est elliptique au sens ci dessus, à coefficients convenablement réguliers. Soit U solution de $L_i U = f_i$, où $f_i \in C_{s_i+t_i, -s_i+t_i}$; soit $|f|_*$ le sup des normes des f_i . On suppose aussi que u_j a des dérivées d'ordre $\leq t_j$ Hölder

continues (exposant α) dans \mathcal{D} , avec $M_{t-t_0,0}(u_f)$ fini. Alors u_f est dans $C_{t-t_0, t_0+\alpha}$ et

$$|u_f|_{t-t_0, t_0+\alpha} \leq K \left(\sum_{t, t_0 > 0} M_{t-t_0,0}[u_f] + |f|^* \right).$$

Dans le cas particulier d'un opérateur elliptique du deuxième ordre ce théorème est dû à Schauder (les A. redémontrèrent simplement ce théorème (§ 3)). La démonstration du cas général est longue et difficile; on commence par le cas des coefficients constants mais vu la très générale notion d'ellipticité introduite, même dans ce cas le résultat n'est nullement évident. Dans le cas particulier des opérateurs différentiels du deuxième ordre, les A. donnent également des évaluations au voisinage de la frontière. Comme application les A. donnent des résultats de différentiabilité des solutions d'équations elliptiques linéaires et non linéaires.

J. L. Lions (Nancy).

Il'in, V. A. Expansion of functions having a singularity in series of eigenfunctions. Kernels of fractional order. Dokl. Akad. Nauk SSSR (N.S.) 105 (1955), 18-21. (Russian)

Let u_i, λ_i be eigen-functions and values for $(\Delta + \lambda)u = 0$ for a finite plane region G with homogeneous boundary conditions. For fixed P inside G and $-2 < \varepsilon < 3/2$, $\varepsilon \neq 0$, let $v_{PQ} = r^\varepsilon - R^\varepsilon$ for $r = r_{PQ} \leq R$, $v_{PQ} = 0$ for $r \geq R$, the circle $r \leq R$ lying inside G . The author finds an asymptotic expansion for $\int_G v u_i d\sigma$ in decreasing powers of λ_i , the leading terms being

$$u_i(P) \{ 2\pi \varepsilon 2^{\varepsilon} \Gamma(\frac{1}{2}\varepsilon) (\Gamma(-\frac{1}{2}\varepsilon))^{-1} \lambda_i^{-1-\varepsilon} + 2\pi \varepsilon R^{\varepsilon} \lambda_i^{-1} J_0(R \sqrt{\lambda_i}) \};$$

the proof is given for $0 < \varepsilon < 3/2$. Modifications of v with smoother behaviour at $r=R$ are also considered. One application is to the expansion of functions with the same singularity as v , another is to prove the existence for $\alpha > 0$ of a kernel $K_\alpha(P, Q)$ whose Fourier development is $\sum u_i(P) u_i(Q) \lambda_i^{-\alpha}$, convergence of the latter series being apparently not involved. Finally he enunciates the extension to N dimensions. [Previous work of the author: same Dokl. (N.S.) 74 (1950), 413-416, 653-656; MR 13, 350.]

F. V. Atkinson (Canberra).

Il'in, V. A. Sufficient conditions for expansibility in an absolutely and uniformly convergent series of eigenfunctions. Dokl. Akad. Nauk SSSR (N.S.) 105 (1955), 210-213. (Russian)

Let C be any piecewise smooth contour inside G [see the preceding review for notation], $h(s)$ a piecewise continuous function defined on C . The author establishes for $\varepsilon > 0$ the absolute convergence of

$$\sum u_i(P) \lambda_i^{-1-\varepsilon} \int_C u_i(s) h(s) ds,$$

and deduces that a function of the form

$$\int_C K_{\frac{1}{2}+\varepsilon}(P, s) h(s) ds + \iint_G K_{\frac{1}{2}+\varepsilon}(P, Q) H(Q) dQ,$$

where $H(Q)$ is of L^2 over G , has a Fourier expansion in the u_i which is absolutely convergent, uniformly in any interior sub-region of G . Suggested applications are to the expansion of functions whose derivatives are discontinuous on C , and to physical problems involving discontinuous media. The N -dimensional analogous are also given.

F. V. Atkinson (Canberra).

Penna, Anna Maria. Sulla verifica delle condizioni al contorno in alcuni problemi relativi ad equazioni a derivate parziali. Univ. e Politec. Torino. Rend. Sem. Mat. 14 (1954-55), 329-369.

The author considers partial differential equations of second order and elliptic type, in two independent variables, with coefficients continuous on and within a circle Γ (radius R , centre the origin of coördinates), and having continuous first and second partial derivatives within the circle. The equation is invariant for rotation about the origin, that is after introduction of polar coördinates ϱ, θ , invariant for substitution of $\theta + \alpha$ for θ . Let $E(U) = 0$ be the equation. A solution is required satisfying certain given conditions on Γ . If the solution is expanded in a Fourier series $U(\varrho, \theta) = \sum_{n=-\infty}^{\infty} c_n(\varrho) e^{in\theta}$, the terms have to satisfy separately an equation which gives rise to an ordinary differential equation for c_n , $L(c_n) = 0$, having a solution $c_n = a_n \gamma_n$, $\gamma_n = \varrho^{|n|} (1 + \varepsilon(\varrho))$, where $\varepsilon(\varrho)$ is continuous and infinitesimal for $\varrho \rightarrow 0$. In Dirichlet's problem U is given on Γ ; hence at a point (Γ, R) , $U = F(\tau)$. If the Fourier expansion of $F(\tau)$ is $\sum_{n=-\infty}^{\infty} h_n e^{in\tau}$, this leads to

$$U(\varrho, \theta) = \sum_{n=-\infty}^{\infty} h_n \frac{\gamma_n(\varrho)}{\gamma_n(R)} e^{in\theta}.$$

To prove that this is indeed the required solution certain points concerning uniform convergence must be considered. In most cases the most difficult one is that, if (ϱ, θ) tends to the point (R, τ) on Γ , $U(\varrho, \theta)$ tends to $F(\tau)$. The author admits that the problem with the same function $F(\tau)$ on Γ has been completely solved for an analogous (simpler) differential equation $E^*(U^*) = 0$. Let the equation be

$$U^*(\varrho, \theta) = \sum_{n=-\infty}^{\infty} h_n \frac{\gamma_n^*(\varrho)}{\gamma_n^*(R)} e^{in\theta}.$$

Consider now

$$U(\varrho, \theta) - U^*(\varrho, \theta) = \sum_{n=-\infty}^{\infty} h_n \frac{\gamma_n(\varrho)}{\gamma_n(R)} - \frac{\gamma_n^*(\varrho)}{\gamma_n^*(R)} e^{in\theta}.$$

If we succeed in proving $\lim_{\varrho \rightarrow R, \theta \rightarrow \tau} \{U(\varrho, \theta) - U^*(\varrho, \theta)\} = 0$, we will have $\lim U(\varrho, \theta) = -f(\tau)$. In the same way the Dirichlet problem for the domain outside Γ may be treated, and the analogous Neumann problems and also mixed problems where on Γ some linear function of U and $\partial U / \partial n$ is given.

As examples of application of this idea the validity of the solutions of the temperature equation $\Delta u = k^2 u$ for a plane circular plate is proved, with the aid of the analogous solutions of the Laplace equation $\Delta u = 0$, and also the more complicated problems for a spherical plate.

The proofs are given with the aid of some inequalities concerning the solutions of the modified Bessel equation $x^2 y'' + xy' + (x^2 - \nu^2)y = 0$.

H. Bremekamp (Delft).

Maurin, K. Bemerkungen über die Methoden von Trefftz und Ritz. Bull. Acad. Polon. Sci. Cl. III. 3 (1955), 573-577.

Comme il est bien connu les problèmes de Dirichlet relatifs à un opérateur différentiel elliptique A d'ordre $2m$ auto-adjoint sur un ouvert Ω de R^n peuvent se ramener au schéma suivant: on munit un espace de Hilbert convenable H de fonctions sur Ω d'une structure hilbertienne associée à A ; soit Z l'adhérence dans H des fonctions à support compact; on décompose $H = Z \oplus U$ (dans la structure considérée). Soit $f \in H$; soit $f = z + u$ la décomposition correspondante; u est solution de $Au = 0$ et vérifie $u - f \in Z$ (ce qui signifiera que u et ses dérivées d'ordre $\leq m-1$

prennent sur la frontière de Ω les „mêmes valeurs” que les dérivées correspondantes de f). Remarques sur l'approximation de u à l'aide de projecteurs orthogonaux sur des sous espaces vectoriels de dimension finie de U (Trefftz) ou de Z (Ritz).
J. L. Lions (Nancy).

★ Garnir, H. G. "Fonctions" de Green pour les problèmes aux limites de l'équation des ondes. Second colloque sur les équations aux dérivées partielles, Bruxelles, 1954, pp. 83-94. Georges Thone, Liège; Masson & Cie, Paris, 1955.

On considère l'opérateur des ondes amorties

$$E = -\Delta + a \frac{\partial^2}{\partial t^2} + b \frac{\partial}{\partial t} + c,$$

où Δ = laplacien, opérant sur un ouvert Ω de R^n , et où a, b, c sont des constantes, $a > 0$. On étudie les problèmes mixtes avec conditions aux limites du type Dirichlet-Neumann (i.e. condition de Dirichlet sur un morceau de la frontière de Ω , de Neumann sur le reste de la frontière). On peut résoudre (au moins formellement) ce type de problème, par une transformation de Laplace en t (Doetsch). Soit donc

$$E(\varphi) = -\Delta + a\varphi^2 + b\varphi + c, \quad \varphi = \xi + i\eta.$$

Pour $\operatorname{Re} \varphi$ assez grand, $E(\varphi)$ admet, relativement au problème de Dirichlet-Neumann considéré, un opérateur de Green $G(\varphi)$. On montre que $G(\varphi)$ est transformée de Laplace en t d'une distribution en t à valeurs dans un espace d'applications linéaires [cf. pour la théorie des distributions à valeurs vectorielles: Séminaire Schwartz, Faculté des Sciences de Paris, 1953-54; MR 17, 764; Grothendieck, Mem. Amer. Math. Soc. no. 16 (1955); MR 17, 763; et, pour l'application actuelle, Lions, les œuvres analysés ci-dessous]. La distribution est définie par un noyau $G_{x,x_0,t}$, le noyau de Green du problème mixte. L'A. montre que, pour toute fonction φ de t indéfiniment différentiable à support compact, $G(\varphi)$ est représentée par un noyau fonction: $G(\varphi) = R(x_0, x; \varphi)$. L'A. donne les propriétés essentielles de $R(x_0, x; \varphi)$ ainsi que ses rapports avec les „fonctions de Green” des physiciens. Applications aux formules de résolution des problèmes mixtes, les problèmes aux limites étant pris dans un sens généralisé. Exemples si $\Omega = R^n$ et dans de nombreux cas où Ω est un domaine polyédral [cf. H. G. Garnir, Bull. Soc. Roy. Sci. Liège 21, 119-140, 207-231, 328-344 (1952); 22, 29-46 (1953); MR 15, 130, 710].

J. L. Lions (Nancy).

★ Lions, Jacques-Louis. Problèmes aux limites de type mixte. Second colloque sur les équations aux dérivées partielles, Bruxelles, 1954, pp. 25-36. Georges Thone, Liège; Masson & Cie, Paris, 1955.

In this paper the author solves the Cauchy problem for the equation

$$\frac{\partial^2 u}{\partial t^2} + u + b \frac{\partial}{\partial t} u + cu + Du = v,$$

where b, c are constants and D a Hermitian or near-Hermitian operator. The Laplace transform is used; the Cauchy data are absorbed into the inhomogeneous term v by permitting v to be a distribution. In application D is a differential operator, its domain restricted to functions satisfying certain boundary conditions; hence the title of the paper.

P. D. Lax (New York, N.Y.).

★ Schwartz, Laurent. Problèmes aux limites dans les équations aux dérivées partielles elliptiques. Second

colloque sur les équations aux dérivées partielles, Bruxelles, 1954, pp. 13-24. Georges Thone, Liège; Masson and Cie, Paris, 1955.

„Exposé du chap. I de la thèse de J. L. Lions” analysée ci-dessous.

Lions, J. L. Problèmes aux limites en théorie des distributions. Acta Math. 94 (1955), 13-153.

La première partie de ce travail (pp. 13-85) constitue une systématique des problèmes aux limites non évolutifs auxquels on peut appliquer les techniques hilbertiennes.

L'auteur présente d'abord une théorie abstraite des problèmes aux limites posés, dans un ouvert Ω de R^n , pour une fonctionnelle bilinéaire gauche définie dans un sous-espace hilbertien d'un certain espace fondamental, en général localement convexe mais, le plus souvent, hilbertien. Il établit l'existence d'une solution unique et stable moyennant une hypothèse d'ellipticité relative à la fonctionnelle considérée.

Ensuite, l'auteur détermine dans quelles conditions il est possible d'attacher à un opérateur différentiel les éléments fondamentaux de son formalisme. Ainsi, la plupart des problèmes aux limites elliptiques étudiés dans la littérature s'incorporent dans le schéma général qu'il préconise.

Pour montrer l'unification réalisée, l'auteur étudie succinctement un certain nombre de problèmes aux limites spéciaux attachés à des opérateurs différentiels elliptiques. Ainsi, il examine les opérateurs du second ordre, dégénérés ou non, à coefficients variables, réguliers ou irréguliers et ce, pour toutes les conditions aux limites usuelles (entre autres, celles de Dirichlet, de Neumann, de Dirichlet sur une partie de la frontière et de Neumann sur l'autre, de Poincaré, de transmission). Il étudie de même les opérateurs différentiels d'ordre $2m$.

La seconde partie du mémoire (pp. 85-153) concerne l'étude des problèmes aux limites d'évolution posés dans le cylindre $\Omega \times]0, \infty[$ de l'espace des x, t . Comme dans la première partie, l'auteur pose ici des problèmes aux limites très généraux, problèmes dont la formulation est rendue possible par le recours aux distributions définies dans un espace de Hilbert ou plus généralement de Banach (notion due à L. Schwartz). Se basant sur la théorie de la transformation de Laplace de ces distributions, l'auteur établit l'existence, l'unicité et la stabilité de la solution des problèmes qu'il propose. Il montre l'intérêt de ses résultats en traitant quelques problèmes particuliers importants. Citons, entre autres, l'équation des télégraphistes généralisée et certaines équations du quatrième ordre. On trouve notamment ici, pour la première fois peut-être, un formalisme permettant de poser et de résoudre les problèmes aux limites de l'opérateur des ondes ou de la diffusion avec un degré de généralité suffisant.

Le mémoire s'achève par des considérations plus particulières. L'auteur examine différents cas dans lesquels les problèmes qu'il propose admettent une solution exprimée par la distribution d'une fonction satisfaisant aux conditions initiales en un sens plus fort. Il y arrive dans le cas des problèmes relatifs à l'équation des ondes et de la diffusion, soit en faisant appel à la théorie des semi-groupes, soit par une méthode directe. Enfin, pour l'équation des ondes, il détermine le domaine d'action des données initiales, indépendamment des conditions aux limites admises; il montre que son résultat ne peut être amélioré, sauf pour des conditions aux limites particulières.

Ce travail apporte dans la théorie des problèmes aux

limites (et spécialement de ceux qui concernent les opérateurs d'évolution) plusieurs idées neuves, importantes par leur simplicité foncière et par leur efficacité. Le lecteur qui s'intéresserait principalement à ces idées en trouve un résumé dans les communications de L. Schwartz, de l'auteur, et du reviewer analysées ci-dessus.

H. G. Garnir (Liège).

Lions, J. L.; et Schwartz, L. Problèmes aux limites sur des espaces fibrés. *Acta Math.* 94 (1955), 155-159.

Extension des résultats de J. L. Lions [cf. l'analyse précédente] pour les opérateurs différentiels sur une variété différentiable. Cas particuliers: les systèmes différentiels dans un ouvert de R^n , étudiés par Lions dans son mémoire et les problèmes aux limites pour les formes différentielles définies dans un espace de Riemann.

H. G. Garnir (Liège).

Lions, J. L. Sur quelques problèmes aux limites relatifs à des opérateurs différentiels elliptiques. *Bull. Soc. Math. France* 83 (1955), 225-250.

L'auteur illustre l'intérêt de ses conceptions relatives aux problèmes aux limites en traitant une nouvelle série d'exemples justiciables de la méthode exposée dans son mémoire des *Acta Mathematica* (cf. les revues ci-dessus) ou tout au moins d'une généralisation de cette méthode, donnée dans la présente note.

Les problèmes aux limites relatifs à l'opérateur Δ retiennent d'abord son attention. Il les résout dans des ouverts particuliers, dits de Soboleff ou de Nikodým, introduits récemment par J. Deny et lui-même [*Ann. Inst. Fourier, Grenoble* 5 (1953-54), 305-370; MR 17, 646]. Il donne également des indications sur les problèmes relatifs à Δ^2 .

Dans son mémoire déjà cité, l'auteur pose, pour l'opérateur $\Delta\Delta + k^2$, six problèmes aux limites caractérisés classiquement par la donnée de deux quelconques des grandeurs u , $\partial u/\partial n$, Δu , $(\partial/\partial n)\Delta u$ sur Ω ; il traite complètement quatre de ceux-ci et laisse de côté le cas des données $\partial u/\partial n$, Δu ou u , $(\partial/\partial n)\Delta u$. On trouve ici l'étude des problèmes laissés en suspens, dans un ouvert de Nikodým.

H. G. Garnir (Liège).

John, Fritz. A note on "improper" problems in partial differential equations. *Comm. Pure Appl. Math.* 8 (1955), 591-594.

On rencontre dans les applications certains problèmes relatifs aux équations aux dérivées partielles, mal posés au sens de J. Hadamard, pour lesquels il est intéressant d'étudier la stabilité (dans un sens à fixer dans chaque cas) de la solution lors de la variation des données, éventuellement dépendantes. L'auteur en donne quelques exemples. Si on restreint a priori la nature des solutions admises pour ces problèmes, on peut obtenir des résultats dans cet ordre d'idées, sans que l'examen des données permette généralement de voir si ces conditions a priori sont satisfaites. L'auteur étudie en détail le cas du laplacien dans un domaine de R^n avec les données de Cauchy sur une partie de la frontière. Ici, la restriction a priori la plus importante consiste à supposer la solution bornée.

H. G. Garnir (Liège).

John, Fritz. Plane waves and spherical means applied to partial differential equations. Interscience Publishers, New York-London, 1955. viii+172 pp. \$4.50. Ce remarquable ouvrage est consacré aux intégrales planes et sphériques et à leurs applications dans différents

chapitres de la théorie des équations aux dérivées partielles. L'exposé présente, sous une forme didactique et unifiée, un grand nombre de résultats obtenus par l'auteur ces dernières années.

La première moitié du livre (ch. I à III) repose sur les formules de J. Radon [*Ber. Verh. Sächs. Ges. Wiss. Leipzig. Math.-Phys. Kl.* 69 (1917), 262-277] donnant l'expression d'une fonction au moyen de ses intégrales sur des hyperplans.

L'auteur traite deux applications importantes de ces formules. Citons d'abord la construction d'une solution élémentaire pour tout opérateur elliptique à coefficients analytique (homogène ou non), l'ellipticité étant définie par le fait que la forme associée aux termes d'ordre maximum est définie positive pour tous les points considérés. L'auteur donne notamment des expressions explicites de ces solutions élémentaires dans le cas des opérateurs à coefficients constants. D'un autre côté, les formules de Radon permettent de résoudre le problème de Cauchy pour les opérateurs $L(\partial/\partial x, \partial/\partial t)$ hyperboliques, homogènes, à coefficients constants, hyperbolique signifiant que l'équation $L(x, t) = 0$ a seulement des racines réelles en t , le coefficient du terme de degré le plus élevé en t ne s'annulant pas. En fait, la question revient à déterminer la solution élémentaire (au sens de la théorie des distributions) des opérateurs considérés. L'auteur étudie également le domaine de dépendance de la solution obtenue, du moins si la dimension de l'espace des x est impaire et inférieure ou égale à l'ordre de l'opérateur diminué de un; autrement dit, il caractérise le support de la solution élémentaire à laquelle nous faisons allusion. On trouve également une étude détaillée du cas particulier de l'opérateur des ondes pures. Signalons que, depuis la parution du présent ouvrage, les formules de J. Radon ont été utilisées pour l'étude du problème de Cauchy pour les opérateurs hyperboliques à coefficients constants non homogènes [cf. l'analyse ci-dessous].

La seconde partie du livre (ch. IV-VIII) est relative aux moyennes sphériques. L'auteur présente d'abord les propriétés essentielles de ces moyennes et en particulier une intéressante formule (qui lui est due) exprimant une moyenne itérée par une intégrale simple de moyenne. Il en déduit l'expression d'une fonction par ses moyennes sphériques.

Débordant du cadre qu'il s'est fixé, l'auteur consacre un chapitre à l'étude des moyennes ellipsoïdales, généralisant le théorème de la moyenne de L. Asgeirsson [*Math. Ann.* 113 (1936), 321-346] par une curieuse identité de A. Howard [*Dissertation, Univ. of Kentucky, 1952*] qui transforme les dérivées secondes d'une moyenne ellipsoïdale en dérivées premières par rapport aux coefficients de l'équation de l'ellipsoïde.

L'auteur revient ensuite aux applications des moyennes sphériques. Ainsi, il traite complètement le problème de la détermination d'une fonction à partir de ses intégrales sur des sphères de rayon fixé, celui de la différentiabilité des équations elliptiques, linéaires ou non, et étudie enfin la régularité de certaines transformées intégrales des solutions d'équations non elliptiques.

L'ouvrage est bien présenté, de lecture attrayante.

H. G. Garnir (Liège).

Courant, R., and Lax, A. Remarks on Cauchy's problem for hyperbolic partial differential equations with constant coefficients in several independent variables. *Comm. Pure Appl. Math.* 8 (1955), 497-502.

Résolution du problème de Cauchy pour un opérateur à

coefficients constants d'ordre k ou un système de k opérateurs du premier ordre, le premier cas seulement étant traité explicitement. Soit $L(\partial/\partial x, \partial/\partial t)$ l'opérateur considéré, d'ordre k , non nécessairement homogène; $x=x_1, \dots, x_n$. On peut, sans restreindre la généralité, se limiter à chercher la solution $u(x, t)$ de $Lu=g(x, t)$, les données initiales étant nulles. Les auteurs réduisent la recherche de $u(x, t)$ au problème de Cauchy pour un opérateur $L^*(\partial/\partial y, \partial/\partial t)$ où $y=\sum_{i=1}^n \alpha_i x_i$, α vecteur paramétrique de module 1 dans R^n , en prenant pour inconnue auxiliaire l'expression

$$\omega^*(y, t) = \int_{R^{n-1}} u(y, y_2, \dots, y_n, t) dy_2 \dots dy_n.$$

On revient de $\omega^*(y, t)$ à $u(x, t)$ en utilisant une formule de J. Radon [cf. l'analyse ci-dessus]. La méthode exige que $L^*(\partial/\partial y, \partial/\partial t)$ soit hyperbolique et que la solution de son problème de Cauchy dépende continuellement de α , ce qui peut ne pas être le cas si L est hyperbolique et admet des caractéristiques multiples, comme le montre un contre-exemple de P. D. Lax annexé à la présente note.

H. G. Garnir (Liège).

Lewy, Hans. Extension of Huyghens' principle to the ultrahyperbolic equation. *Ann. Mat. Pura Appl.* (4) 39 (1955), 63-64.

Let \square denote the ultrahyperbolic operator defined by

$$\square u = \sum_{i=1}^3 \frac{\partial^2 u}{\partial x_i^2} - \sum_{i=1}^3 \frac{\partial^2 u}{\partial y_i^2},$$

and consider the non-homogeneous equation $\square u = f(x, y)$. If a is positive, and if $r^2 = \sum_{i=1}^3 x_i^2$, $s^2 = \sum_{i=1}^3 y_i^2$, then the surface C whose equation is $r+s=a$ is a characteristic surface of this equation. Position on C is specified by a parameter t , such that $r=t$, $s=a-t$, and angle variables on the spheres $r=\text{constant}$, y fixed and $s=\text{constant}$, x fixed.

From the fact that the integral over C of $rs \square u$ is equal to

$$\int_0^a dt \iint d\omega_1 \iint d\omega_2 \left(\frac{\partial^2}{\partial r^2} - \frac{\partial^2}{\partial s^2} \right) (rsu),$$

where $d\omega_1$ and $d\omega_2$ are the elements of solid angle in the x and y spaces, it follows that

$$\int_0^a dt \iint d\omega_1 \iint d\omega_2 rs f(x, y) = \frac{4}{3} \pi a \left\{ \iint u_1 d\omega_1 - \iint u_2 d\omega_2 \right\}$$

where $u_1 = u$ with $r=a$, $s=0$ and $u_2 = u$ with $r=0$, $s=a$. This is the generalised Asgerisson formula of spherical averages. In the proof, the function u is required to satisfy the equation $\square u = f(x, y)$ only on the characteristic manifold $r+s=\text{constant}$. E. T. Copson (St. Andrews).

Diaz, J. B., and Ludford, G. S. S. On a theorem of Le Roux. *Canad. J. Math.* 8 (1956), 82-85.

Dans un ouvert R du plan (x, y) on donne l'opérateur

$$Lu = u_{xy} + a(x, y)u_x + b(x, y)u_y + c(x, y)u$$

à coefficients continus; soit $U(x, y, \alpha)$ une famille de solutions de $L_{xy}u=0$, avec $\tilde{U}(x, y, \alpha) = U(x, y, \alpha)/(x-\alpha)^2$, $0 < \alpha < 1$, $(x, y) \in R$, $\alpha_0 \leq \alpha < x$, \tilde{U} , \tilde{U}_x , \tilde{U}_y , \tilde{U}_{xy} , $\tilde{U}_{\alpha x}$, $\tilde{U}_{\alpha y}$, $\tilde{U}_{\alpha xy}$ continues dans $(x, y) \in R$, $\alpha_0 \leq \alpha < x$. Alors

$$(a) \quad u(x, y) = \int_{\alpha_0}^x (x-\alpha)^{-1} \tilde{U}(x, y, \alpha) f(\alpha) d\alpha$$

est solution de $Lu=0$ quelle que soit f une fois continu-

ment différentiable. Pour la démonstration, il faut d'abord intégrer une fois (a) par parties, puis dériver sous le signe somme.

J. L. Lions (Nancy).

O'Keeffe, J. The initial value problem for the wave equation in the distributions of Schwartz. *Quart. J. Mech. Appl. Math.* 8 (1955), 422-434.

Résolution du problème de Cauchy pour l'équation des ondes $-\Delta + (1/c^2)(\partial^2 u/\partial t^2) = g$ avec $\Delta = \sum_{i=1}^n \partial^2 u/\partial x_i^2$, en faisant appel aux éléments de la théorie des distributions de L. Schwartz [Théorie des distributions, t. I, II, Hermann, Paris, 1950, 1951; MR 12, 31, 833]. L'auteur détermine d'abord la solution élémentaire de l'opérateur étudié en calculant la transformée de Fourier inverse par rapport aux variables d'espace (et au sens des distributions) de la solution élémentaire de l'opérateur

$$(1/c^2)(\partial^2 u/\partial t^2) + \sum_{i=1}^n \xi_i^2.$$

Posant le problème de Cauchy généralisé comme le fait Schwartz, [loc. cit., t. I, p. 131], il le résout par composition avec la solution élémentaire et retrouve ainsi la forme explicite des formules résolitives connues.

H. G. Garnir (Liège).

Pounder, J. R., and Synge, J. L. Note on the initial-value problem for the wave equation in N dimensions. *Proc. Roy. Irish Acad. Sect. A.* 57 (1955), 151-159.

Nouvelle méthode de résolution du problème de Cauchy pour l'équation des ondes $\sum_{i=1}^n \partial^2 u/\partial x_i^2 - \partial^2 u/\partial t^2$, avec $n > 1$. Le point de départ est une solution complexe de cette équation dépendant de deux points a et b de l'espace des x , it , non singulière si $\beta \neq 0$. Introduite dans la formule de Green relative à la région de R^{n+1} déterminée par le plan $t=C$ passant par a et une surface σ située en-dessous de ce plan, cette solution particulière permet d'obtenir une certaine expression $K(u, \beta)$ en fonction des valeurs de u et de du/dn sur σ . L'expression $K(u, \beta)$ tend vers $u(a)$ à un multiple près, lorsque β tend vers 0. Passant à la limite, les auteurs trouvent la solution du problème de Cauchy étudié; la valeur de $u(a)$ obtenue s'avère ne dépendre que de celles de u et de du/dn sur la section de σ par le cône caractéristique rétrograde issu de a , si n est pair, par la surface de ce cône, si n est impair. Le procédé n'est pas sans élégance mais paraît limité à l'équation étudiée.

H. G. Garnir (Liège).

Kline, Morris. Asymptotic solutions of Maxwell's equations involving fractional powers of the frequency. *Comm. Pure Appl. Math.* 8 (1955), 595-614.

Considérons, dans R^3 , un problème aux limites donné pour les équations de Maxwell, E et H désignant respectivement les champs électriques et magnétiques. Soient $E_0(x, t)$, $H_0(x, t)$ la solution de ce problème déterminée par une distribution de charge de densité $g(x)H(t)$ [$H(t)$: fonction de Heaviside]. D'autre part, si la densité de charge est $g(x) \exp(-i\omega t)$, les champs électromagnétiques sont représentés par $E(x, t) = u(x) \exp(-i\omega t)$, $H(x, t) = v(x) \exp(-i\omega t)$ après établissement de l'état stationnaire. Dans le présent mémoire, on déduit le développement asymptotique de u [resp. v] selon les puissances éventuellement fractionnaires de $1/\omega$, sachant que pour x donné, E_0 [resp. H_0] ne présente que des discontinuités par saut brusque au voisinage desquelles on connaît son allure. A chaque discontinuité de E_0 [resp. H_0] correspond une série asymptotique partielle de u [resp. v] et la représentation asymptotique cherchée est la somme de ces

dernières. Ce travail perfectionne d'autres notes du même auteur [mêmes Comm. 4 (1951), 225-262; J. Rational Mech. Anal. 3 (1954), 315-342; MR 13, 408; 15, 800].

H. G. Garnir (Liège).

Krzyżański, Mirosław. Sur la solution fondamentale de l'équation aux dérivées partielles du type parabolique. Ann. Mat. Pura Appl. (4) 40 (1955), 89-97.

Let the fundamental solution of the equation

$$(*) \quad a(x, y)u_{xx} = u_y + b(x, y)u_x + c(x, y)u$$

be written in the form $\sum_{n=0}^{\infty} U_n(x, y; \xi, \eta)$. The iterative procedure used by W. Feller [Math. Ann. 113 (1936), 113-160] to determine U_{n+1} made use of U_n , $\partial U_n / \partial y$, and U_0 ; while J. Hadamard [C. R. Acad. Sci. Paris 152 (1911), 1148-1149] used only U_n and U_0 to determine U_{n+1} for the equation

$$(**) \quad u_{xx} = u_y + c(x, y)u.$$

The author points out that by making the appropriate transformations one can find the fundamental solution of equation (*) using the method proposed by Hadamard. The restrictions placed on the coefficients of (*) by Feller are not as stringent as those mentioned in the present paper.

F. G. Dressel (Durham, N.C.).

Pini, Bruno. Sulla regolarità e irregolarità della frontiera per il primo problema di valori al contorno relativo all'equazione del calore. Ann. Mat. Pura Appl. (4) 40 (1955), 69-88.

Conditions for the existence of the solution of the Dirichlet problem for Laplace's equation $\Delta u = 0$ and a given domain have been stated in terms of capacities, barriers, and conductor potentials [O. D. Kellogg, Foundations of potential theory, Springer, Berlin, 1929]. The author proposes the use of similar concepts and functions in connection with the first boundary-value problem for the parabolic equation $L = u_{xx} - u_y = 0$ and a given domain D . (This problem will be referred to as problem $L(D)$). In an earlier paper [Rend. Sem. Mat. Univ. Padova 23 (1954), 422-434; MR 16, 485; see this review for terminology] he defines and points out some of the uses of parabolic barrier functions. In the present paper necessary and sufficient conditions for the existence of the solution of problem $L(D)$ are given in terms of a so-called parabolic conductor potential of the domain D . To each domain the author associates a number which he calls the parabolic capacity of the domain. In terms of parabolic capacities he gives a set of necessary and also a set of sufficient conditions for the existence of a solution of problem $L(D)$.

F. G. Dressel (Durham, N.C.).

Ding, Shia-Shi. Differential equations of mixed type. Acta Math. Sinica 5 (1955), 193-204. (Chinese. English summary)

For the mixed-type equation (1): $(\text{sign } y)u_{xx} + (\text{sign } x)u_{yy} = 0$, the following "Tricomi" problem is found to have a unique solution. § 1. In the (x, y) -plane take the points $A(0, 0)$, $B(1, 0)$, $C(0, 1)$, $D(\frac{1}{2}, -\frac{1}{2})$ and $E(-\frac{1}{2}, \frac{1}{2})$, and the segments AD , AE , DB and DE . Connect B and C by a continuously differentiable simple arc σ which lies in the first quadrant. In the domain G which is bounded by σ and the four segments, find a solution of (1) with the properties: 1) u takes on prescribed values on σ , AD and AE where the assigned functions $\varphi(s)$, $\varphi_1(x)$ and $\varphi_2(x)$ are of class C_1 , 2) in G , u is of class C_2 except along the coordinate axes, where the continuity of u_x and u_y is required,

3) u is continuous on the closure of G (including the vertices). The solution is obtained by an application of a method of Lavrent'ev and Biczadze [Dokl. Akad. Nauk SSSR (N.S.) 70 (1950), 373-376; MR 11, 724]. In § 2-§ 4 the Cauchy problem with data on $y=0$, $x>0$, and the problem with data $u(x, 0)=\tau(x)$, $u(x, -x)=\varphi(x)$, are solved for the equation (2): $y u_{xx} + x u_{yy} = 0$ in the fourth quadrant of the (x, y) -plane. These solutions are obtained by transformations to the Euler-Poisson equation by well-known procedures. It appears from the text that § 2-§ 4 are intended as preliminary work on "Tricomi" problems for equation (2).

Y. W. Chen.

Ou, Sing-Mo; et Ding, Shia-Shi. Sur l'unicité du problème de Tricomi de l'équation de Chaplygin. Acta Math. Sinica 5 (1955), 393-399. (Chinese. French summary)

Consider the equation (1): $K(y)u_{xx} + u_{yy} = 0$, where $K(0)=0$, $yK(y) \geq 0$, and $K'(y) \geq 0$, in a domain D which consists of two parts E and H . E lies in the upper half $(y>0)$, and H in the lower half $(y<0)$ plane, with E being bounded from above by a curve σ , and H bounded from below by two characteristics OB and AB ; E and H have a segment OA on the x -axis as the common boundary. Let the distance of the two vertical supporting lines of E be l , and let $y=-b$ at the point B . Introduce the expression $F(y) = 2(K/K')' + 1$, where $'$ denotes differentiation with respect to y . The authors prove the following uniqueness theorem: a solution of (1) with Tricomi data is unique in D , if there is a non-negative constant β such that

$$\max_{b \leq y \leq 0} K' \cdot F(y) / (2K\sqrt{-K}) \leq \beta \leq 2\pi/l$$

holds. This extends the uniqueness theorem by F. I. Frankl, in which $F(y) \geq 0$ was assumed (and hence $\beta=0$), by admitting negative values for $F(y)$. The proof is a variant of an argument used by Protter [J. Rational Mech. Anal. 2 (1953), 107-114; MR 14, 654]. Now since $F(0)=3$, there is a first zero of odd order at $y=-1/\alpha$ for $F(y)$. If $F(y)<0$ for $y<-1/\alpha$, and if in addition $\int_0^{1/\alpha} (F(y)-1)dy \leq 0$ holds, then, as is shown by the authors, there is a value $y=y_0 < -1/\alpha$, where $K(y)$ becomes infinite. Thus restrictions on l and b are needed in general for the uniqueness.

Y. W. Chen (Berkeley, Calif.).

Moskvičev, A. D. On a particular solution of the equation $\Delta^* \Delta^* w(x, y) = f(x, y)$ for a half-strip. Kulbyšev. Indust. Inst. Sb. Nauč. Trud. 1953, no. 4, 13-19. (Russian)

The solution of the interior boundary-value problem for the semi-infinite strip $0 \leq x \leq \infty$, $0 \leq y \leq b$, defined by the equations $\Delta^* \Delta^* w(x, y) = f(x, y)$, $w = w_{yy} = 0$ on $y=0$ and $y=b$, $w = w_{xx} = 0$ on $x=0$ and in the limit as $x \rightarrow \infty$, is referred to the solution of a problem considered earlier by the author [Kulbyšev. Indust. Inst. Sb. Nauč. Trud. 1950, 41-46]. The earlier problem, not stated in the present paper, presumably treats the equation $\Delta^* u(x, y) = f(x, y)$ for the same strip. Here Δ^* is the generalized Laplacian operator in the sense of Privaloff.

R. N. Goss (San Diego, Calif.).

De Giorgi, Ennio. Un teorema di unicità per il problema di Cauchy, relativo ad equazioni differenziali lineari a derivate parziali di tipo parabolico. Ann. Mat. Pura Appl. (4) 40 (1955), 371-377.

Let $[y]$ denote the largest integer not superior to y , and let R denote the rectangle $a_1 \leq x \leq a_2$, $t_1 \leq t \leq t_2$. Con-

sider the boundary-value problem

$$\frac{\partial^m u}{\partial t^m} = \sum_{h=0}^{m-1} \sum_{k=0}^{[a(m-1)]} c_{hk}(x, t) \frac{\partial^{h+k} u}{\partial x^k \partial t^h} \quad (0 < \alpha < 1),$$

$$\frac{\partial^h u}{\partial t^h} = 0, \text{ on } t=t_1, a_1 \leq x \leq a_2, h=0, 1, \dots, m-1.$$

The c_{hk} are assumed to be continuous on R and to have all derivatives with respect to x . The author shows that the only solution to the above boundary-value problem is $u=0$ in case there is a positive constant ϱ such that

$$\lim_{n \rightarrow \infty} \frac{\varrho^n}{n!} \left| \frac{\partial^n c_{nk}}{\partial x^n} \right| = 0 \quad (0 \leq h \leq m-1; 0 \leq k \leq \alpha(m-n)).$$

F. G. Dressel (Durham, N.C.).

See also: Fourier, p. 698; Lee, p. 727; Sneddon, p. 732; Guseinov, p. 750; Fichera, p. 770; Hartman and Wintner, p. 781; Tersenov, p. 803; Nakata and Fujita, p. 804; Tonolo, p. 808.

Integral Equations

Reuter, G. E. H. Über eine Volterrasche Integralgleichung mit totalmonotonem Kern. Arch. Math. 7 (1956), 59-66.

Given a kernel $k(u)$, positive on $[0, +\infty)$ and summable on compacts, it is not a simple matter to decide whether the bounded continuous solution to

$$f(t) + \int_0^t f(t-u)k(u)du = 1 \quad (t > 0)$$

is positive or not. One sufficient condition is that k be small in the sense that $\int_0^\infty k(u)du < 1$. Here, it is shown that the solution is not only positive but completely monotonic whenever k is completely monotonic. This fact is important for the construction of certain special Markov chains: see A. N. Kolmogorov [Moskov. Gos. Univ. Uč. Zap. 148, Mat. 4 (1951), 53-59; MR 14, 295]; D. G. Kendall and G. E. H. Reuter [Proc. Symposium on Stochastic Processes, Amsterdam, 1954 (in press)].

H. P. McKean, Jr. (Princeton, N.J.).

Certkov, I. Ya.; and Haskind, M. D. On a class of multidimensional integral equations of Volterra type. Mikolaiv. Derž. Ped. Inst. Nauk. Zap. 1953, no. 4, 136-146. (Ukrainian)

Multidimensional integral equations of the Volterra type are solved by the use of multidimensional Laplace transforms. The inversion of the multidimensional transform is accomplished by means of the Riemann-Mellin integral formula in which the integration is carried out over a multidimensional complex region. Equations of the type solved here occur in the study of supersonic flow around thin airfoils. A particular case considered is a two-dimensional analogue of Abel's equation.

H. P. Thielman (Ames, Iowa).

Grebenyuk, D. G. On a method of approximate solution of Fredholm integral equations. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. 15 (1955), 107-110. (Russian)

The author considers the approximate solutions of Fredholm's integral equations,

$$\varphi(x) = \lambda \int_a^b K(x, s) \varphi(s) ds + f(x) \quad (a \leq x, s \leq b),$$

in the form of a generalized polynomial $P_n(x) = \sum \varphi_i \varphi_i(x)$.

The polynomial $P_n(x)$ is constructed so that the difference $f(x) - \sum \varphi_i \varphi_i(x)$, where

$$b_i(x) = \varphi_i(x) - \lambda \int_a^b K(x, s) \varphi_i(s) ds,$$

has the least deviation from zero in the interval (a, b) .

S. Kulik (Columbia, S.C.).

Stesin, I. M. Application of continued fractions to finding the solution of integral equations. Dokl. Akad. Nauk SSSR (N.S.) 105 (1955), 225-228. (Russian)

If $G_n(x)$ is the characteristic polynomial of a finite matrix A , and if $\lambda = x^{-1}$ is not an eigenvalue, then the equation $y - \lambda A y = b$ is solved by the Sylvester formula

$$(1) \quad y = x(xI - A)^{-1}b = \frac{x}{G_n(x)} \left[\frac{G_n(xI) - G_n(A)}{xI - A} \right] b.$$

The purpose of this note is to announce and prove an extension of (1) when A is a selfadjoint completely continuous operator in Hilbert space, corresponding to a symmetric, positive definite kernel $K(x, s)$. [See the review of Stesin, Vychisl. Mat. Vychisl. Tehn. 2 (1955), 145-150; MR 17, 414.] We wish to solve

$$(2) \quad y(x) = \lambda \int_a^b K(x, s) y(s) ds + f_0(x) = \lambda A y + f_0,$$

where $f_0 \in L_2$. Let $P_n(-Z)/Q_n(-Z)$ be the n th convergent of (*), loc. cit., and let $Z_{n1} > Z_{n2} > \dots$ be the roots of $Q_n(-Z) = 0$. Theorems: I. The sequence of functions

$$y_n(x) = \frac{x}{Q_{2n}(-x)} \left[\frac{Q_{2n}(-x) - Q_{2n}(-A)}{xI - A} \right] f_0(x)$$

converges uniformly in x to the solution $y(x)$, as $n \rightarrow \infty$, if $\lambda = x^{-1}$ is not an eigenvalue of (2). II. The sequence

$$\phi_{ni}(x) = (Z_{ni} - A)^{-1} Q_{2n}(-A) f_0(x) R_{2n}(-Z_{ni})$$

converges uniformly in x to the i th normed eigenfunction of the kernel $K(x, s)$, as $n \rightarrow \infty$. Here

$$R_n(Z) = (P_n(Z) Q_n'(Z))^{1/2}.$$

G. E. Forsythe (Los Angeles, Calif.).

Evgrafov, M. A. The spectral theory of operators of a certain form in the space of analytic functions. Dokl. Akad. Nauk SSSR (N.S.) 105 (1955), 625-627. (Russian)

Let $A(r)$ be the space of functions analytic in $|z| < r$, $B(r)$ the space of functions analytic in $|z| > r$ (with the topology of uniform convergence on compact subsets). The author considers an integral operator $L(F) = (2\pi i)^{-1} \int_{|z|=r} K(z, \zeta) F(\zeta) d\zeta$, where

$$K(z, \zeta) = \sum_{n=0}^{\infty} \lambda_n z^n \zeta^{-n-1} + \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \epsilon_{nk} z^{k+n} \zeta^{-n-1}.$$

It is assumed that all λ are different and $\lambda_n \rightarrow 0$ and that $\lim_{n \rightarrow \infty} \sum_{k=0}^{\infty} |\epsilon_{nk}| r^{k-n} = 0$ for $0 < c < r < 1$. There is a biorthogonal system $\{\phi_n(z), \psi_n(\zeta)\}$ associated with $K(z, \zeta)$ where $\phi_n(z) = z^n + a_1^{(n)} z^{n+1} + \dots$ and $\psi_n(\zeta) = \zeta^{-n} + b_1^{(n)} \zeta^{-n-1}$ are power series in z and $1/\zeta$ respectively,

$$\lambda_n \phi_n(z) = (2\pi i)^{-1} \int_{|z|=r} K(z, \zeta) \phi_n(\zeta) d\zeta,$$

$$\lambda_n \psi_n(\zeta) = (2\pi i)^{-1} \int_{|z|=r} K(z, \zeta) \psi_n(z) dz.$$

A condition is given under which $\{\phi_n(z)\}$ is a basis of $A(r)$ and $\{\psi_n(\zeta)\}$ is a basis of $B(r)$ ($c < r < 1$). W. H. J. Fuchs.

Ickovič, I. A. Inversion of Giraud's formula. Kišinev. Gos. Univ. Uč. Zap. 11 (1954), 7-11. (Russian)
From the formula

$$\varphi(\lambda) = - \int_{\lambda-\pi}^{\pi} \log(\sin(\lambda-\psi)) (f(\psi) + f(\psi+\pi)) d\psi \\ + \frac{1}{2} \pi \int_{\lambda-\pi}^{\lambda} (f(\psi) - f(\psi+\pi)) d\psi$$

assuming that $f(\psi)$ satisfies a Lipschitz condition and that $\int_0^{2\pi} f(\psi) d\psi = 0$ the author derives the formula

$$f(\varphi) = - \frac{1}{4\pi^2} \int_0^{\pi} \cot(\varphi-\lambda) \frac{\partial}{\partial \lambda} (\varphi(\lambda+\pi) + \varphi(\lambda)) d\lambda \\ - \frac{1}{4\pi^2} \frac{\partial}{\partial \varphi} (\varphi(\varphi+\pi) - \varphi(\varphi)).$$

D. C. Kleinecke (Albuquerque, N.M.).

Freidkin, S. A. Solution of a class of singular integral equations. Kišinev. Gos. Univ. Uč. Zap. 11 (1954), 13-17. (Russian)

Consider the singular integral equation

$$f(x) = a\varphi(x) + (c\pi)^{-1} \int_L (t-x)^{-1} \varphi(t) dt,$$

where L is the union of an infinite set of disjoint intervals on the real line. Consider the space of functions whose modulus squared is summable over L with respect to the weight

$$p(x) = (- \prod_{k=1}^{\infty} [(a_k-x)/(b_k-x)])^{\frac{1}{2}}.$$

If L is bounded and $a^2 \neq b^2$, then the equation has a unique solution in the space considered. *D. C. Kleinecke.*

Gohberg, I. C. On systems of singular integral equations. Kišinev. Gos. Univ. Uč. Zap. 11 (1954), 55-60. (Russian)

The results of an earlier paper [Uspehi Mat. Nauk (N.S.) 7 (1952), no. 2(48), 149-156; MR 14, 54] are extended to systems of singular integral equations.

D. C. Kleinecke (Albuquerque, N.M.).

Bykov, Ya. V. On the theory of linear integro-differential equations of Volterra's type. Kirgiz. Gos. Univ. Trudy Fiz.-Mat. Fak. 1953, no. 2, 67-83. (Russian)

Let $[c, d]$ be an interval containing a and x ; $a_{ij}(x)$ and $K_{ij}(x, t)$ functions continuous for x and t in $[c, d]$. The system

$$(1) \quad \frac{dz_i}{dx} + \sum_{j=1}^n a_{ij}(x) z_j(x) - \lambda \int_a^x \sum_{j=1}^n K_{ij}(x, t) z_j(t) dt, \\ z_i(x_0) = b_i \quad (i=1, 2, \dots, n)$$

has, for every λ , a unique solution $z_i(x)$ if $x_0 = a$. This solution can be obtained from a fundamental system of solutions as in the case of systems of linear differential equations. However, the Wronskian of this system can vanish in points $x_0 \neq a$. A consequence of this is that (1) does not have a solution for arbitrary λ and b_i if $x_0 \neq a$. For a sequence of λ values which are the characteristic numbers of an equivalent Fredholm equation (1) has a solution (infinitely many) only for special b_i .

M. Golomb (Lafayette, Ind.).

Egorov, A. I. Existence theorem for the solution of an integro-differential equation. Kirgiz. Gos. Univ. Trudy Fiz.-Mat. Fak. 1953, no. 2, 119-123. (Russian)

By the usual method of successive approximations the

author proves that the equation

$$\frac{dy}{dx} = f(x, y) + \lambda \int_a^b K(x, s) f_1(s, y(s)) ds$$

has a unique integral curve through every point of a certain rectangle $a \leq x \leq b$, $c \leq y \leq d$ if $f(x, y)$, $f_1(x, y)$ as functions of y are Lipschitzian. *M. Golomb.*

Guseinov, M. G. On a boundary problem for certain differential and integro-differential equations. Azerbaidžan. Gos. Univ. Trudy. Ser. Fiz.-Mat. 4 (1954), 61-73. (Russian. Azerbaijani summary)

The equations considered are

$$\Delta^2 U = f(x, y, z, U, \Delta U)$$

and

$$\Delta^2 U = f(x, y, z, U, W_1, W_2, \dots, W_n),$$

where $W_i(P) = \int_{|Q|=r} K_i(P, Q; U(Q)) d\tau$, $P = P(x, y, z)$, $Q = Q(\xi, \eta, \zeta)$, $|Q| = (\xi^2 + \eta^2 + \zeta^2)^{\frac{1}{2}}$; the boundary conditions are $U(P) = dU(P)/dn = 0$ for $|P| = r$. These problems are transformed into equivalent integral equations and systems of integral equations by the aid of the known Green's function for this case. Under various hypotheses concerning $f(P, u, u_1)$, $f(P, u, u_1, u_2, \dots, u_n)$, $K_i(P, Q, u)$, of which the crucial one is that these functions be Lipschitzian as to the u 's, the author proves with the aid of Schauder's fixed-point theorem that these equations have a unique solution for sufficiently small r . There are many distracting misprints in this paper, e.g., "f" for "j" in (6) and many following equations; " $\partial/\partial t$ " for " $\partial/\partial v$ " in the equation preceding (18). *M. Golomb.*

Guseinov, A. I. On a class of nonlinear integral equations. Akad. Nauk Azerbaidžan. SSR. Trudy Inst. Fiz. Mat. 4-5 (1952), 20-23. (Russian. Azerbaijani summary)

The author proves that the equation $u(x) = f(x) + \int_0^x F(x, s, u(s)) ds$ has a solution $u(x)$ for $0 \leq x \leq \delta$ to which the successive approximations

$$u_{n+1}(x) = f(x) + \int_0^x F(x, s, u_n(s)) ds$$

converge provided F satisfies certain conditions. The most important of these are: $F(x, s, u)$ is positive, increasing in u , and $F(x, s, u)/u \rightarrow 0$ as $u \rightarrow \infty$. Additional conditions insure uniqueness of the solution. *M. Golomb.*

Guseinov, A. I. On the theory of nonlinear singular integral equations. Akad. Nauk Azerbaidžan. SSR. Trudy Inst. Fiz. Mat. 3 (1948), 57-64. (Russian. Azerbaijani summary)

In an earlier paper [Izv. Akad. Nauk SSSR. Ser. Mat. 12 (1948), 193-212; MR 9, 443] the author studied singular integral equations of the form (1) $u(x) = \lambda v(x)$, where $v(x) = \int_a^b (s-x)^{-1} K(x, s, u(s)) ds$. In the present paper he obtains analogous results for the equation (2) $u(x) = \lambda f(x, u, v)$. If K satisfies the conditions of the earlier paper and if

$$|f(x_1, u_1, v_1) - f(x_2, u_2, v_2)| \leq A_1 |x_1 - x_2|^{\delta} + A_2 |u_1 - u_2| \\ + A_3 |v_1 - v_2| \quad (0 < \delta < 1),$$

then equation (2) has a solution in class H [for the definition of H see the earlier review]. If, moreover, f_u, f_v satisfy conditions analogous to that satisfied by K_u in the earlier paper, then the solution of (2) is unique and can be obtained by successive approximations.

M. Golomb (Lafayette, Ind.).

Guseinov, A. I. On an application of nonlinear singular integral equations. Akad. Nauk Azerbaldžan. SSR. Trudy Inst. Fiz. Mat. 6 (1953), 14-24. (Russian. Azerbaijani summary)

In this paper the author extends the results of the note reviewed above to systems of the form

$$u_i(x) = f_i(x, u_1, \dots, u_n; v_1, \dots, v_n) \quad (i=1, \dots, n),$$

where $u_i(x)$ are the unknown functions and $v_i(x)$ are the singular (principal-value) integrals $\int_a^b (x-s)^{-1} K_i(x, s, u_i(s)) ds$. *M. Golomb.*

Smirnov, M. M. On singular solutions of nonlinear integral equations. Vestnik Leningrad Univ. 9 (1954), no. 11, 3-17. (Russian)

Consider the equation

$$(*) \quad \varphi(x) = \lambda \int_0^1 K(x, t) f(t, \varphi(t)) dt,$$

where kernel $K(x, t)$ is definite, symmetric, and continuous, and $f(t, z) = \sum_{n=0}^{\infty} A_n(t) z^n$, and $A_n(t)$ is continuous for $0 \leq t \leq 1$. The existence of singular solutions of (*) is investigated, i.e., solutions $\varphi(x, \lambda)$ of (*) in a neighborhood of $\lambda=0$ such that $\lim_{\lambda \rightarrow 0} \varphi(x, \lambda) = \infty$. The method consists of substituting a formal series

$$(**) \quad \varphi(x) = (\lambda_0/\lambda) \varphi_{-1}(x) + \varphi_0(x) + \lambda \varphi_1(x) + \lambda^2 \varphi_2(x) + \dots$$

in (*), equating like powers of λ , and solving the resulting system of integral equations to determine λ_0 and $\varphi_n(x)$ ($n=-1, 0, 1, 2, \dots$). The study of the system of integral equations must be divided into a number of cases. For some cases, a singular solution of the form (**) is obtained. For others, negative fractional powers of λ must be introduced. The case $m=2$ is considered in detail, and the results for arbitrary m are indicated. *J. Cronin.*

Satō, Tokui; et Iwasaki, Akira. Sur l'équation intégrale de Volterra. Proc. Japan Acad. 31 (1955), 395-398.

The paper deals with the Volterra integral equation

$$(1) \quad u(x) = f(x) + \int_0^x K(x, t, u(t)) dt,$$

where $f(x)$ is bounded and measurable in the interval $I: 0 \leq x \leq 1$, and $K(x, t, u)$ is defined in the domain D given by $0 \leq t \leq x \leq 1, -\infty < u < \infty$. Moreover, the latter function is continuous in x and u for almost all t in I , measurable with respect to t and has a summable function of t as a majorant. Among the results proved we mention the following.

i) If K is non-decreasing in u , then there exists a maximal solution of (1) in I , i.e., a solution $U(x)$ such that $U(x) \geq u(x)$ for every solution $u(x)$ of (1). Let $\bar{f}(x)$, $\bar{K}(x, t, u)$ satisfy the same assumptions as f and K . Let $\bar{f}(x) \leq f(x)$ and (2) $K(x, t, u) \leq \bar{K}(x, t, \bar{u})$ for $u < \bar{u}$, and let $\bar{u}(x)$ be the maximal solution of the equation (1') obtained from (1) by replacing f and K by \bar{f} and \bar{K} respectively. Then $u(x) \leq \bar{u}(x)$ for every bounded and measurable solution $u(x)$ of (1). ii) In addition to (2) let

$$|K(x, t, u) - K(x, t, u')| < \bar{K}(x, t, |u - u'|).$$

If then equation (1') with $\bar{f}=0$ has only the solution $u=0$, then the solution of (1) is unique. *E. H. Rothe.*

Vainberg, M. M. Topological methods of investigation of a problem on eigenfunctions of nonlinear integral equations. Moskov. Oblast. Pedagog. Inst. Uč. Zap. Trudy Kafedr Mat. 20 (1954), 37-57. (Russian)

This paper deals with the existence of nontrivial so-

lutions for

$$(1) \quad u(x) = \lambda \int_B \dots \int_B K(x; y_1, y_2, \dots, y_n) \times g(u(y_1), \dots, u(y_n); y_1, \dots, y_n) dy_1 \dots dy_n$$

and similar equations when $g(0, \dots, 0; y_1, \dots, y_n) = 0$. The main tool is a lemma which in slightly different form was proved by Rothe [Amer. J. Math. 66 (1944), 245-254; MR 6, 71]: If the completely continuous operator $F(x)$ with domain and range in the cone K of a Banach space satisfies $\|F(x)\| \geq a > 0$ for $x \in K$ and $\|x\| = r$, then there exist $x_0 \in K$, $\|x_0\| = r$ and $\lambda_0 > 0$ such that $F(x_0) = \lambda_0 x_0$. By its use the existence of nontrivial solutions $u(x)$ of (1) in C , L_1 and L_p ($p > 1$) is proved under various hypotheses on K and g . *M. Golomb (Lafayette, Ind.).*

Vainberg, M. M. An integral equation of Uryson. Moskov. Oblast. Pedagog. Inst. Uč. Zap. Trudy Kafedr Mat. 21 (1954), 49-64. (Russian)

In the integral equation $\varphi(x) = \lambda \int_B K(x, y, \varphi(y)) dy$ assume $K(x, y, 0) = 0$, $K(x, y, u)$ differentiable with respect to u ,

$$0 < h \leq Q(x, y) \leq K_u'(x, y, u) \leq P(x, y) < H$$

$$\text{for } u > 0 \text{ and } (x, y) \in B \times B,$$

and all occurring functions measurable in $B \times B$, B bounded. Let α, β be the smallest eigenvalues of P, Q respectively. Then the above equation has a unique positive solution $\varphi(x, \lambda)$ for every λ between α and β (and only for these values); $\varphi(x, \lambda)$ is a continuous monotone increasing function of λ and $\lim_{\lambda \rightarrow \alpha} \varphi(x, \lambda) = 0$, $\lim_{\lambda \rightarrow \beta} \varphi(x, \lambda) = \infty$. These results, which are slightly sharper than those obtained originally by Uryson [Mat. Sb. 31 (1923), 236-255] are derived by the author using Uryson's successive-approximation procedure slightly modified.

M. Golomb (Lafayette, Ind.).

Dubrovskii, V. M. On systems of nonlinear integral equations. Moskov. Gos. Univ. Uč. Zap. 155, Mat. 5 (1952), 206-209. (Russian)

Using the Schauder fixed-point theorem the author proves that the system

$$u_i(x) = f_i(x) + \lambda \int_B K_i(x, y, u_1(y), \dots, u_n(y)) dy$$

has a solution for all real λ and continuous $f_i(x)$ if the K_i satisfy certain conditions, in particular,

$$|K_i(x, y, u_1, \dots, u_n)| \leq A(1 + \sum_{j=1}^n |u_j|^{q_j})$$

with $0 < q_j < 1$.

M. Golomb (Lafayette, Ind.).

Voskresenskii, E. P. On nonlinear integral equations of Hammerstein type. Voronež. Gos. Univ. Trudy. Fiz.-Mat. Sb. 27 (1954), 75-78. (Russian)

The author obtains some of the results concerning existence of nontrivial solutions known for the equation $\lambda \varphi(x) = \int_B K(x, y) g(\varphi(y), y) dy$ with symmetrical kernel for the case where $K(x, y) = A(x)B(y)S(x, y)$ with $0 < a < A(x) < b$, $0 < c < B(y) < d$ and $S(x, y)$ symmetric. *M. Golomb.*

Kal'nibolackaya, L. A. On complete continuity of the operator $Au = \int_G K(x, y, f(y, u(y))) dy$. Vestnik Leningrad. Univ. 9 (1954), no. 8, 71-78. (Russian)

Suppose the integral operator

$$Au = \int_G K[x, y, f(y, u(y))] dy$$

satisfies the following conditions. Let G be a bounded closed domain in Euclidean n -space, S a set of functions $u(x) \in L^2(G)$ such that $|u(x)| < u_0(x)$ where $u_0(x)$ is given function, $0 \leq u_0(x) \leq \infty$. Assume $f(x, y)$ continuous in $x \in G$ and $-\infty < y < \infty$, $[f(x, y)]^2 \leq L(x)$ for $|y| \leq u_0(x)$, where $L(x) \in L^1(G)$, and that the operator $f[x, u(x)]$ takes $L^2(G)$ into itself; $K(x, y, z)$ continuous in $x, y \in G$ and $[K(x, y, z)]^2 \leq M(x, y)$ for $|z| \leq \sqrt{L(y)}$ where $M(x, y) \in L^1(G \times G)$. Then $A(S) \subset L^2(G)$ and A is completely continuous on S . If, in addition, $u_0(x) \in L^2(G)$, then for all λ such that $|\lambda| \leq \inf_{x \in G} u_0(x) / \int_G \sqrt{M(x, y)} dy$, the equation $\varphi(x) = \lambda \int_G K[x, y, f(y, \varphi(y))] dy$ has a solution $\varphi^*(x)$ and $|\varphi^*(x)| \leq u_0(x)$. Analogous theorems are obtained for a system of integral equations. *J. Cronin.*

Ahmedov, K. T. The method of degenerate kernels for a class of equations in the space L_2 . *Izv. Akad. Nauk Azerbaidžan. SSR.* 1953, no. 4, 27-34. (Russian. Azerbaijani summary)
Suppose the kernels of the equation

$$(1) \quad u(x) + \int_0^1 \sum_{j=1}^m \lambda_j K_j(x, y) f_j(y, u(y)) dy = 0$$

are in L_2 , symmetric and positive definite; the functions $f_j(y, u)$ are in L_2 for $|u| \leq L$ and $|f_j(y, u_2) - f_j(y, u_1)| \leq M_j |u_2 - u_1|$ where $M_j < \lambda_j$, λ_j being the smallest characteristic number of K_j . Then equation (1) has a unique solution $u(x)$ in L_2 if $\sum M_j \lambda_j^2 / \lambda_j < 1$. The solution is obtained as the limit (in L_2) of the solutions of the approximating algebraic systems that result from (1) when the K_j are replaced by the partial sums of their bilinear expansions. *M. Golomb* (Lafayette, Ind.).

Ahmedov, K. T. Existence theorems for the solution of certain systems of nonlinear equations. *Azerbaidžan. Gos. Univ. Trudy. Ser. Fiz.-Mat.* 4 (1954), 34-36. (Russian. Azerbaijani summary)

This paper contains the proofs to the lemmas used in the paper reviewed above. *M. Golomb.*

Jost, Res. Mathematical analysis of a simple model for the stripping reaction. *Z. Angew. Math. Phys.* 6 (1955), 316-326.

The author discusses a simplified model of stripping reaction which leads to a generalization of an integral equation of the Wiener-Hopf type. A procedure for solving this integral equation is discussed. *A. E. Heins.*

See also: Gahov and Čibrikova, p. 722; Blackwell, p. 754; Sobolev, p. 769; Freldkin, p. 769; Krasnosel'skiĭ, p. 769; Babuška, p. 791; Vasil'ev, p. 792; Bahtin and Krasnosel'skiĭ, p. 803; Chakrabarty and Gupta, p. 805.

Calculus of Variations

★ **Grüss, Gerhard.** *Variationsrechnung.* 2te Aufl. herausgegeben von W. Meyer-König. Quelle & Meyer, Heidelberg, 1955. viii+282 pp. DM 14.00.

This monograph is principally devoted to the classical conditions for extrema of simple integrals in the plane. For the most part admissible curves are required to be of class C' , and this requires use of a lemma on the rounding of corners. However, there is a section on discontinuous solutions in ch. VI, which treats problems in parametric form. The derivation of the conditions (both necessary and sufficient) is rather close to the historical order of

development. Ch. VII gives a brief discussion of the first necessary condition for problems involving higher-order derivatives, problems in more than two dimensions, isoperimetric problems (where normality is explicitly assumed), the problem of Lagrange with finite side conditions, and problems involving double integrals. The editor of this revised edition, W. Meyer-König, remarks that he has modified the derivation of the multiplier rule for isoperimetric problems. He has also expanded the former Section 26 on direct methods into a new ch. VIII, which includes remarks on existence proofs, and minimizing sequences, on relations to characteristic value problems, and on the method of Ritz. The chapter still seems to be too brief to be of much use to the student. The book as a whole seems to be quite carefully written as an introduction to the subject. *L. M. Graves.*

Bellman, Richard; Glicksberg, Irving; and Gross, Oliver. Some nonclassical problems in the calculus of variations. *Proc. Amer. Math. Soc.* 7 (1956), 87-94.

The authors treat a number of special problems of which the following is typical. Choose $f(t)$ satisfying the constraints $0 \leq f \leq M > 1$, $\int_0^T f dt \leq \alpha < T$ so as to minimize $J(f) = \int_0^T |1 - x(t)| dt$, where $x(t)$ is the absolutely continuous solution of the differential equation $dx/dt = -x + f$, $x(0) = 1$. The basic device is to convert the problem into the game

$$\min_t \max_\phi \int_0^T \phi(1-x) dt$$

with $\phi(t)$ any measurable function such that $-1 \leq \phi(t) \leq 1$ on $(0, T)$. The minimizing f_0 turns out to be the characteristic function of an interval $(0, a)$.

W. H. Fleming (Lafayette, Ind.).

See also: Busemann, p. 779.

Theory of Probability

Bohm, D.; and Schützer, W. The general statistical problem in physics and the theory of probability. *Nuovo Cimento* (10) 2 (1955), supplemento, 1004-1047.

The authors do not consider that the theory of probability is either sufficiently flexible or necessary for some of the needs of physics. In general, they stress the conceptual significance of situations in which the weighting assigned to events is asymptotically independent of the weighting assigned to the determining physical parameters. In probability language, this means that certain limiting probability distributions for large times are independent of the distributions of these parameters. This point of view has also been stressed recently by Khintchine [In memory of Aleksandr Alexandrovič Andronov, *Izdat. Akad. Nauk SSSR, Moscow*, 1955, pp. 541-574; MR 17, 567]. It usually appears that the physical parameters must be excluded from exceptional sets of zero measure (which may however be dense in terms of the natural topology). The authors consider that, even if a system is in a state determined by such an exceptional parameter value, it will shift to a desirable state because of environmental disturbances. It appears to the reviewer, but obviously not to the authors, that this approach is at most a simple rule determining the basic probability measures to which the usual probability calculus can then be applied, with the understanding that the conclusions must still be checked experimentally. *J. L. Doob.*

Motzkin, T. S. The probability of solvability of linear inequalities. Proceedings of the Second Symposium in Linear Programming, Washington, D.C., 1955, pp. 607-611. National Bureau of Standards, Washington, D.C., 1955.

The "probability" that m linear equations in n unknowns have a solution is clearly 1 for $m \leq n$ and 0 for $m > n$. On the other hand, even for $m > n$ there is a finite probability (intuitively speaking) that m inequalities in n unknowns have a solution. Theorem: For any reasonable definition of probability (this phraseology is the reviewer's, not the author's) the probability $p(m, n)$ that m inequalities in n unknowns, $m > n$, have a solution, is given by $p(m, n) = 2^{-m} \sum_{k=0}^m \binom{m}{k}$. Proof: "For any reasonable definition of probability" the occurrence of the inequalities $\sum_{i=1}^m a_i x_i \leq a_0$ or $\sum_{i=1}^m a_i x_i \geq a_0$ is "equally likely". Now the m bounding hyperplanes determined by the m inequalities are in general position with probability 1, hence they divide n -space up into $\sum_{k=0}^m \binom{m}{k}$ polyhedral regions (well-known). Hence, of the 2^m equally likely ways of choosing the signs for the inequalities, exactly $\sum_{k=0}^m \binom{m}{k}$ ways yield a solvable system.

D. Gale (Providence, R.I.).

Hunt, G. A. An inequality in probability theory. Proc. Amer. Math. Soc. 6 (1955), 506-510.

Let x, y be random variables in R_1 and $\mathcal{E}(x)$ the mathematical expectation of x . Then y is said to be dominated by x if $\mathcal{E}[\varphi(x)] \geq \mathcal{E}[\varphi(y)]$ for every convex function. The following lemma is proved: Let $\mathcal{E}(y) = 0$ and let $|y| \leq 1$ with probability one. Then y is dominated by every symmetric random variable x such that $1 \leq |x| < \infty$. By the help of this lemma a generalization of an inequality of Khinchin is immediately obtained. An application is also made to matrices $Y = (y_{ij})$ with independent random variables, which are dominated by Gaussian variables x_{ij} . An estimation of $\Pr\{\|Y\| \geq a\}$ is given, $\|Y\|$ being the largest absolute value of any characteristic value of Y .

H. Bergström (Göteborg).

Ram, Siya. Multidimensional hypergeometric distribution. Sankhyā 15 (1955), 391-398.

Given a population containing $N = N_1 + N_2 + \dots + N_{k+1}$ units, N_i being the number of units of the i th kind. The k -dimensional hypergeometric distribution is defined as the joint distribution of the number of occurrences of items of the $k+1$ kinds if n of the N units are selected without replacement. The paper gives recurrence relations for the moments and cumulants of the k -dimensional hypergeometric distribution, as well as formulas for the factorial moments. The recurrence relations are used to find moments and cumulants for $k=1$ and 2.

G. E. Noether (Boston, Mass.).

Zinger, A. A.; and Linnik, Yu. V. On an analytic generalization of a theorem of Cramér's and its application. Vestnik Leningrad. Univ. 10 (1955), no. 11, 51-56. (Russian)

Generalizing the Cramér theorem that, if the sum of two independent random variables is Gaussian, the summands are, the authors prove that, if f_1, \dots, f_s are characteristic functions, and if, for some positive constants $\alpha_1, \dots, \alpha_s$, and real γ ,

$$\prod_j f_j^{2\alpha_j}(t) = e^{i\gamma t - \beta t^2}$$

in a neighborhood of $t=0$, then f_j is the characteristic function of a Gaussian distribution. This result is then applied to derive a simple proof of Skitovič's theorem [Dokl. Akad. Nauk SSSR (N.S.) 89 (1953), 217-219; MR 14, 1098] that, if X_1, \dots, X_n are independent random variables, and if $\sum_j a_j X_j$ and $\sum_j b_j X_j$ are independent, then X_j is Gaussian whenever $a_j b_j \neq 0$. J. L. Doob.

Sapogov, N. A. The problem of stability for a theorem of Cramér's. Vestnik Leningrad. Univ. 10 (1955), no. 11, 61-64. (Russian)

Let X_1, X_2 be independent random variables with sum X . Let X have distribution function F , suppose that X_1 has median 0, and suppose that $\sup_x |F(x) - \Phi(x)| \leq \epsilon < 1$, where Φ is the normal distribution function of mean 0 and variance 1. Then

$$\sup_x \left| F_1(x) - \Phi\left(\frac{x-a_1}{\sigma_1} \right) \right| < \frac{C}{\sigma_1^3 (\ln \epsilon^{-1})^{\frac{1}{2}}}$$

Here C is an absolute constant, a_1 and σ_1^2 are the mean and variance of X_1 after X_2 is replaced by 0 for values of the modulus $\geq N$, where N depends on ϵ , and it is supposed that $\sigma_1 \neq 0$. This result, whose proof is sketched, improves a previous generalization of the Cramér theorem by the author [Izv. Akad. Nauk SSSR. Ser. Mat. 15 (1951), 205-218; MR 13, 51]. If the random variables are n -dimensional vector-valued, the following weaker result is stated. If R is an n -dimensional interval with edges parallel to the coordinate axes, suppose that

$$\sup_R |P\{X \in R\} - G(R)| \leq \epsilon < 1,$$

where G is some non-degenerate normal distribution. Then, if X_1 has a non-degenerate distribution, there is a normal distribution G_1 , satisfying

$$\sup_R |P\{X_1 \in R\} - G_1(R)| \leq C \left(\ln \ln \frac{1}{\epsilon} \right) \left(\ln \frac{1}{\epsilon} \right)^{-1/(n+1)}.$$

Here C is a measure of the non-degeneracy of X_1 .

J. L. Doob (Urbana, Ill.).

Petrov, V. V. On precise estimates in limit theorems. Dokl. Akad. Nauk SSSR (N.S.) 104 (1955), 180-182; errata 106 (1956), 582. (Russian)

Announcement of an improvement on Linnik's results [Izv. Akad. Nauk SSSR. Ser. Math. 11 (1947), 111-138; MR 8, 591] on the accuracy of the normal approximation in the classical framework of Lyapunov-Cramér-Berry-Esseen. Linnik's condition about small probability near zero of the distributions (with mean zero) is omitted and in the symmetrical case examples show the result is best possible even up to the numerical factor. Details can await full publication. K. L. Chung (Syracuse, N.Y.).

Petrov, V. V. On precise estimates in limit theorems. Vestnik Leningrad. Univ. 10 (1955), no. 11, 57-58. (Russian)

Let X_1, X_2, \dots be mutually independent random variables with zero expectations. Let \bar{F}_n be the distribution function of $(\sum_1^n X_j)/S_n$, where S_n^2 is the variance of the numerator, and let Φ be the normal distribution function (expectation 0, variance 1). Conditions are given sufficient that, together with the hypothesis of symmetry of the X_j distributions,

$$|\bar{F}_n(x) - \Phi(x)| < \frac{C_0}{(2\pi n)^{\frac{1}{2}}} e^{-\frac{1}{2}x^2} (1 + \epsilon_n(M)), \quad |x| < M.$$

The conditions are imposed on the second and third moments, and truncated moments, of the X_j 's. The constant C_0 is evaluated in terms of the stated conditions, and $\varepsilon_n(M) \rightarrow 0$ with $1/n$ in a way independent of everything but the imposed conditions. If the symmetry hypothesis is weakened to the hypothesis that $p \leq P\{X_j > 0\} \leq 1-p$, for some positive p , the conclusion is modified as follows. The multiplier C_0 will now depend on p , and $\bar{F}_n - \Phi$ is replaced by $\bar{F}_n - \Phi - H_n$, where

$$H_n(x) = \frac{1-x^2}{6(2\pi)^{1/2}} e^{-1/2x^2} \frac{1}{S_n^3} \sum_{j=1}^n \int_{-\infty}^{\infty} x^3 dP\{X_j \leq x\}.$$

These results refine earlier results of Linnik [Izv. Akad. Nauk SSSR. Ser. Mat. 11 (1947), 111-138; MR 8, 591].
J. L. Doob (Urbana, Ill.).

Shapiro, J. M. Error estimates for certain probability limit theorems. *Ann. Math. Statist.* 26 (1955), 617-630.

The author considers a system of random variables

$$\{x_{nk}: k=1, 2, \dots, k_n; n=1, 2, \dots\}$$

such that for each $n \geq 1$ the variables $(x_{n1}, x_{n2}, \dots, x_{nk_n})$ are mutually independent; he writes S_n for the sum of this last group of random variables and F_n for its distribution function, and recalls that if

$$(*) \quad \max_{1 \leq k \leq k_n} \text{pr}\{|x_{nk}| > \varepsilon\}$$

tends to zero as n tends to infinity, for each $\varepsilon > 0$, then the class of possible limiting distributions of F_n coincides with the class of infinitely divisible distributions F . Now let F be an infinitely divisible distribution with mean μ and variance σ^2 and with

$$(**) \quad i\mu t + \int_{-\infty}^{\infty} (e^{itx} - 1 - itx) \frac{1}{x^2} dG(x)$$

as the logarithm of its characteristic function; let $x_{nk} - \mathcal{E}(x_{nk})$ satisfy (*), let $F_n \rightarrow F$ at all continuity points of the latter, and let $\text{var}(S_n) \rightarrow \sigma^2$. Let (***) F have a bounded derivative and let $\text{var}(x_{nk}) \leq 1$ (all n, k). Then the author gives bounds for $\sup_{-\infty < x < \infty} |F_n(x) - F(x)|$ in the spirit of A. C. Berry [Trans. Amer. Math. Soc. 49, (1941), 122-136; MR 2, 228] and C-G. Esseen [Acta Math. 77 (1945), 1-125; MR 7, 312], and shows that his bounds approach zero when $n \rightarrow \infty$.

The results simplify considerably when G of (**) is simple in form, and so in particular when F is Gaussian or Poisson; even so they are too complicated for summary here. Other theorems are given in which (***) is dropped but in which the F_n 's are required to be step-functions.

D. G. Kendall (Oxford).

Lebedinceva, O. K. On limiting distributions for normalised sums of independent random quantities. *Dopovidi Akad. Nauk Ukrain. RSR* 1955, 12-15. (Ukrainian. Russian summary)

Let ξ_1, ξ_2, \dots be a sequence of independent random variables each of which has a distribution function out of s given different ones. It is conjectured by Gnedenko that the class of all possible limit distributions of normalised sums of the ξ_n consists of all possible compositions of not more than s stable laws. This is proved here under certain rather complicated conditions.
K. L. Chung.

Hewitt, Edwin, and Zuckerman, Herbert S. Arithmetic and limit theorems for a class of random variables. *Duke Math. J.* 22 (1955), 595-615.

Let G be a commutative finite semi-group such that there exists an integer $m > 1$ for which $x^{m+1} = x$ for all $x \in G$. Let, for the complex linear space $F(G)$ of all complex-valued functions $f(x)$, a product be introduced by the "convolution" $f * g(x) = \sum_{uv=x} f(u)g(v)$. By virtue of the theory of Fourier transforms, defined through the "semi-character" of G , the authors discuss the arithmetic and limit theorems in the positive cone

$$P(G) = \{f/f(x) \geq 0 \text{ and } \sum_{x \in G} f(x) = 1\}$$

of $F(G)$. They prove, for instance, (i) $f \in P(G)$ is idempotent if and only if the carrier $C(f)$ of f is a subgroup of G and $f(x) = \text{constant}$ on $C(f)$, and (ii) $f \in P(G)$ is representable as an infinite product if and only if there is a subsemigroup A of G containing a unit and of order at least 2 and a non-void subset D of G such that $C(f) = DA$. The latter result extends the theorem proved by N. N. Vorob'ev [Mat. Sb. N.S. 34(76) (1954), 89-126; MR 15, 882] for the case in which G is a commutative finite group.
K. Yosida (Tokyo).

Blackwell, David. On transient Markov processes with a countable number of states and stationary transition probabilities. *Ann. Math. Statist.* 26 (1955), 654-658.

Let x_n denote the state at the n th epoch of a system whose motion is governed by a "constant" Markov chain with a countable infinity of states, the matrix of one-step transition probabilities being $p_{ij} = \text{pr}\{x_1 = j | x_0 = i\}$. [Note that the author writes this as $p(j|i)$.] Call a group C of states "almost closed" when

$$(a) \quad \text{pr}\{x_n \in C \text{ i.o. (infinitely often)}\} > 0$$

and

$$(b) \quad \text{pr}\{x_n \in C \text{ i.o.} \& x_n \notin C \text{ i.o.}\} = 0.$$

The author shows that there exists an essentially unique sequence C_1, C_2, \dots of disjoint almost closed groups of states such that: (i) with at most one exception the C_k are "atomic" (do not contain disjoint almost closed subsets); (ii) the non-atomic C_k , if present, contains no atomic subsets; (iii) the system is almost certain to enter and thereafter remain in one of the groups of states C_k .

Call such a Markov chain "simple" when there is just one term in the sequence C_1, C_2, \dots . It may be atomic or non-atomic (an example of each situation is given). The author shows that the system of linear equations

$$(*) \quad y_j = \sum_i p_{ij} y_i$$

has no non-constant bounded solution if and only if the Markov chain is simple and atomic.

A number of applications are made to randomwalk processes. The main tool in the proofs is the martingale convergence theorem. This paper contains the first serious attack on the structure of the set of solutions to (*) (the structure of the set of solutions to the equation of which (*) is the adjoint is, of course, well-known). A further analysis of the non-atomic C in relation to (*) would now be timely.
D. G. Kendall (Oxford).

Dynkin, E. B. On new analytic methods in the theory of Markov random processes. *Vestnik Leningrad. Univ.* 10 (1955), no. 11, 69-74. (Russian)

Expository paper, concentrating on semi-group methods.
J. L. Doob (Urbana, Ill.).

Chung, K. L. Some new developments in Markov chains. Trans. Amer. Math. Soc. 81 (1956), 195-210.

Let $\{x_t: t \geq 0\}$ be a countable Markov chain with states $(i: i \geq 0)$ and stationary transition probabilities $p_{ij}(t) = \Pr(x_{t+s} = j | x_s = i)$ such that $p_{ii}(t) \rightarrow 1 (t \downarrow 0, i \geq 0)$ and recall that in these circumstances

$$t^{-1}[1 - p_{ii}(t)] \rightarrow -p_{ii}'(0) \leq +\infty (t \downarrow 0, i \geq 0).$$

Let i be stable in the sense that $p_{ii}'(0) > -\infty$, let $\Pr(x_0 = i) = 1$, and let $\sup\{t: x_s = i, s \leq t\}$ be called sojourn time. The author shows that the post-exit probabilities $\Pr(x_{t+s} = j | \text{sojourn time} = s)$ have continuous versions $r_{ij}(t)$ such that

$$(1) \quad p_{ij}(t) = \int_0^t r_{ij}(t-s) d[1 - e^{p_{ii}'(0)s}] + p_{ij}(0+) e^{p_{ii}'(0)t},$$

in which $e^{p_{ii}'(0)t} = \Pr(\text{sojourn time} > t)$;

$$(2) \quad \sum_j r_{ij}(t) = 1 (t > 0);$$

$$(3) \quad r_{ij}(t+s) = \sum_k r_{ik}(t) p_{kj}(s) (t, s > 0);$$

$$(4) \quad \lim_{t \uparrow +\infty} r_{ij}(t) = \lim_{t \uparrow +\infty} p_{ij}(t),$$

in which the existence of the second limit is due to P. Lévy [Ann. Sci. Ecole Norm. Sup. (3) 68 (1951), 327-381; MR 13, 959]. One sees at once that

$$(1) \Rightarrow (1)_0 \quad p_{ij}'(t) = -p_{ii}'(0)[r_{ij}(t) - p_{ij}(t)] (t > 0);$$

$$(2) \Rightarrow (2)_0 \quad \sum_j p_{ij}'(t) = 0 (t > 0);$$

$$(3) \Rightarrow (3)_0 \quad p_{ij}'(t+s) = \sum_k p_{ik}'(t) p_{kj}(s) (t, s > 0);$$

$$(4) \Rightarrow (4)_0 \quad p_{ij}'(t) \rightarrow 0 (t \uparrow +\infty).$$

Statement $(1)_0$ shows that each $p_{ij}'(t)$ is continuous ($t > 0$), provided only that i be stable, which, together with $(2)_0$ and $(3)_0$, was first proved by D. G. Austin [Proc. Nat. Acad. Sci. U.S.A. 41 (1955), 224-226; MR 16, 1130] by purely analytic means.

These are the main results. Other topics studied are the post-exit process $\{x(t + \text{sojourn time}): t \geq 0\}$ and the embedded renewal processes $\{(\text{number of times } x_s \text{ enters } j, s \leq t): t \geq 0\}$ associated with stable states j . Various results due to J. L. Doob [Trans. Amer. Math. Soc. 52 (1942), 37-64; 58 (1945), 455-473; MR 4, 17; 7, 210] and P. Lévy [loc. cit.] are sharpened in passing. Two misprints were noted: in l. 15 on p. 198, the second u should read $u-s$, and, in l. 27 on p. 206, the i should be a j .

H. P. McKean, Jr. (Princeton, N.J.).

Harris, T. E. On chains of infinite order. Pacific J. Math. 5 (1955), 707-724.

Consider a stationary stochastic process $Z_n (n=0, \pm 1, \dots)$ taking the values $0, 1, \dots, D-1$. Let $Q_i(u) = P(Z_n = i | Z_{n-1} = u_1, Z_{n-2} = u_2, \dots)$. Introduce $X_n = \sum_{j=1}^n Z_{n-j} D^{-j}$. After making precise the relation between the Z and the X processes it is shown that a unique Z process with prescribed Q_i exists provided the latter satisfy certain conditions. This extends a result of Doeblin and Fortet [Bull. Soc. Math. France 65 (1937), 132-148]. Next it is shown that the distribution of X_n has one of three forms, provided certain general "mixing" conditions hold. Finally grouped Markov chains are considered and it is shown how to determine the corresponding $Q_i(u)$ and a similar transition probability given Z_{n-1}, \dots, Z_{n-k} .

K. L. Chung (Syracuse, N.Y.).

Raevskii, S. Ya. On some typical nonlinearities for continuous random events. Vestnik Moskov. Univ. 10 (1955), no. 12, 37-47 (1956). (Russian)

Stochastic functions $X_2(t) = f(X_1(t))$ are considered, where $X_1(t)$ is a Gaussian process, for three special odd polygonal functions $f(x_1)$ of the real variable x_1 . The first- and second-order characteristic functions of $X_2(t)$ are computed, and expressions for the expectation and autocorrelation are given explicitly. The computation is along similar lines to those of Middleton [Quart. Appl. Math. 5 (1948), 445-498; MR 9, 362]. E. Reich.

Tamari, Dov. Une contribution aux théories modernes de communication: machines de Turing et problèmes de mot. Synthèse 9 (1954), 205-227.

Texte d'une conférence destinée à un public de non mathématiciens lors d'un colloque sur la théorie de l'information.

Dans la première partie l'auteur nous dit que la transmission d'idées exactes, même lorsqu'elle se fait "sans bruit" pose paradoxalement des problèmes insolubles: ces problèmes apparaissent quand on communique sur la notion de communication elle-même.

Dans la seconde partie, l'auteur expose avec une grande clarté la notion de machine de Turing-Post et montre l'impossibilité de résoudre le problème des mots pour une classe de semi-groupes libres préordonnés qui s'adapte facilement à la dite machine. Il montre comment passer de là au cas des demi-groupes abstraits et pour finir trace en quelques lignes une esquisse des méthodes de Marshall Hall et de Tartakowski.

Il semble au rapporteur que c'est l'emploi du terme de "communication" là où le mot "codage" conviendrait et la raideur de la transition entre la première et la seconde partie, qui aurait pu être adoucie par quelques explications relatives à l'étroite parenté, si non l'identité, entre les notions de codage, d'algorithme et de machine, qui rend la première partie un peu obscure. J. Riguet (Paris).

Matschinski, Matthias. I fenomeni di fluttuazione in geofisica. Loro descrizione matematica e loro applicazione dal punto di vista pratico. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 18 (1955), 378-385.

★ Arfwedson, Gerhard. Research in collective risk theory. Summary. Thesis, University of Stockholm, 1955. Almqvist & Wiksells Boktryckeri AB, Uppsala, 1955. 23 pp.

This is a summary of the author's thesis prepared for its disputation. The material is covered in the author's recent papers [Skand. Aktuarietidskr. 37 (1954), 191-223; 38 (1955), 37-100; MR 17, 275, 638]. E. Lukacs.

See also: Moiseev, p. 738; Dunin-Barkovskii and Smirnov, p. 755; Van Brocklin, p. 793; Chakrabarty and Gupta, p. 805.

Mathematical Statistics

★ Dunin-Barkovskii, I. V., i Smirnov, N. V. Teoriya veroyatnostei i matematicheskaya statistika v tekhnike. Obščaya čast'. [The theory of probability and mathematical statistics in engineering. General part.] Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1955. 556 pp. 25.85 rubles.

This book on probability and mathematical statistics

is written for engineers and engineering students. Therefore only those parts of probability theory are discussed which are needed for the exposition of the methods of mathematical statistics, and more complicated proofs are frequently omitted. Chapter I (pp. 11-15) gives a brief historical survey which consists mostly of a listing of names and dates of prominent scientists working in the field. The role of the Russian School is strongly emphasized but the work of western statisticians is also briefly discussed (of contemporary scientists R. A. Fisher, J. Neyman and A. Wald are mentioned). In the authors' opinion the results of English and American mathematical statisticians are applied to further reactionary aims. However this view does not prevent the use and exposition of these results in the book. The student of the book will acquire a good working knowledge of statistical methods regardless of their origin.

Chap. II (pp. 16-50), "Basic concepts of probability theory", Ch. III (pp. 51-104), "Random variables and their characteristics", and Ch. IV (pp. 105-177), "Some special distributions", provide an introduction into probability theory which is adequate for the purpose of the book. Chap. V (pp. 178-291), "Sampling methods and statistical estimates of distributions", discusses briefly descriptive statistics and then gives an introduction into the mathematical methods of statistics. The topics discussed include: desirable properties of estimates, methods to obtain point estimates, interval estimation, order statistics. Chap. VI (pp. 292-363), "Testing of statistical hypotheses", contains the usual material but treats the analysis of variance only very briefly. Tests of goodness of fit (including Kolmogorov's test) as well as tests for randomness and tests for normality are discussed. The concept of the power function is introduced in the last paragraph of this chapter. Chap. VII (pp. 364-430), "Fundamentals of Correlation Theory", deals not only with measures of association, estimation problems and testing of hypotheses concerning correlation and regression coefficients but also with least-squares theory. Chap. VIII (pp. 431-490), "Some applications of statistical and probabilistic methods in engineering", treats some specific engineering applications, the precision of measurements and statistical quality control. The authors state here that the idea to apply statistical and probabilistic methods to problems of quality control occurs already in a paper (inaccessible to this reviewer) by M. V. Ostrogradskii [Bull. Cl. Phys.-Math. Acad. Imp. Sci. St.-Petersbourg 6 (1846), no. 21-22, 321-346 (1848)]. The Appendix (pp. 492-550) contains 30 statistical tables (with examples and references to the main body of the text) and a bibliography. The F -table has the misleading headings "degrees of freedom of greater (resp. smaller) mean square" which contradicts the description of the one-tailed test given in Chapter VI. The bibliography contains only books published in Russian and gives no references to the periodical literature. *E. Lukacs.*

Laha, R. G. On some properties of the normal and gamma distributions. *Proc. Amer. Math. Soc.* 7 (1956), 172-174.

The author proves the following theorems: Let x_1, \dots, x_n be n independently and identically distributed random variables and assume that their common distribution is normal [resp. a gamma distribution]. The sum $x_1 + x_2 + \dots + x_n$ and a function $g(x_1, x_2, \dots, x_n)$ are independently distributed if, and only if, $g(x_1, \dots, x_n)$ and $g(x_1 + a, \dots, x_n + a)$ [resp. $g(x_1 a, \dots, x_n a)$] are identically distributed.

The proof is given only for the normal case since the theorem for the gamma distribution follows from an almost identical reasoning. *E. Lukacs.*

Makabe, Hajime; and Morimura, Hidenori. A normal approximation to Poisson distribution. *Rep. Statist. Appl. Res. Un. Jap. Sci. Engrs.* 4 (1955), 37-46.

An estimate of the error term in the normal approximation to the Poisson distribution is given. Their methods make use of earlier work, especially that of C. G. Esseen [Acta Math. 77 (1945), 1-125; MR 7, 312]. A few tabulated values for their error term and for the error term due to T. T. Cheng [Bull. Amer. Math. Soc. 55 (1949), 396-401; MR 10, 613] are given. *M. Muller* (Ithaca, N.Y.).

Makabe, Hajime. A normal approximation to binomial distribution. *Rep. Statist. Appl. Res. Un. Jap. Sci. Engrs.* 4 (1955), 47-53.

An estimate of the error term [similar to that given by Uspensky [Introduction to mathematical probability, McGraw-Hill, New York, 1937] in the normal approximation to the Binomial distribution is given [see also W. Feller, Ann. Math. Statist. 16 (1945), 319-329; MR 7, 459]. The methods used are analogous to those found in the paper reviewed above. *M. Muller.*

Kunisawa, Kiyonori; Makabe, Hajime; and Morimura, Hidenori. Notes on the confidence band of population distribution. *Rep. Statist. Appl. Res. Un. Jap. Sci. Engrs.* 4 (1955), 54-56.

The authors express a few brief expository remarks concerning their experiences with computations related to the Kolmogorov limiting distribution. *M. Muller.*

Anderson, T. W. On estimation of parameters in latent structure analysis. *Psychometrika* 19 (1954), 1-10.

The latent structure discussed is that of a group of persons who answer only "yes" or "no" to K questions of a questionnaire. This group is composed of m subgroups, with proportion v^a ($a=1, 2, \dots, m$) persons in the a th subgroup. Within each subgroup the responses to the questions are independent, i.e., if the probability is λ_i^a that a person taken at random from subgroup a responds "yes" to question i ($i=1, 2, \dots, K$), then the probability λ_{ij}^a of a yes answer to both i and j questions is $\lambda_i^a \lambda_j^a$. The probabilities for the entire group are denoted by π 's with subscripts corresponding to the various response patterns. These probabilities are functions of the v 's and λ 's; for example, $\pi_i = \sum_a v^a \lambda_i^a$; $\pi_{ij} = \sum_a v^a \lambda_i^a \lambda_j^a$ for $i \neq j$, etc. The probabilities π 's can be estimated by observing the response pattern of the individuals. By means of matrix algebra the author inverts the equations for the probabilities of response patterns to only $2m-1$ items to obtain the v 's and λ 's in terms of π 's. He discusses without proof the use of this procedure as an estimation method when the estimates of π 's are substituted for actual π 's. An artificial numerical example is given.

O. P. Aggarwal (Lafayette, Ind.).

Gibson, W. A. An extension of Anderson's solution for the latent structure equations. *Psychometrika* 20 (1955), 69-73.

The author presents Anderson's solution [see the preceding review] of the latent-structure problem in abbreviated form and shows two ways to extend it so that the response pattern probabilities of all items will be used. He states two least-squares properties of one of these extensions. *O. P. Aggarwal* (Lafayette, Ind.).

Kellerer, Hans. Eine Verallgemeinerung des einfachen Urnenmodells und ihre Anwendung in der Stichprobentheorie. *Allg. Statist. Arch.* 39 (1955), 221-226.

Siraždinov, S. H. A simple statistical acceptance control. *Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh.* 15 (1955), 41-56. (Russian)

The author derives an acceptance sampling plan and gives formulae (without derivations) and tables for obtaining unbiased estimates for the number of defective articles accepted. The case of destructive as well as non-destructive test (with replacement of items found defective) is considered. The work is based on a paper by A. Kolmogorov [*Izv. Akad. Nauk SSSR. Ser. Mat.* 14 (1950), 303-326; *MR* 12, 116; 15, 452]. *E. Lukacs.*

Lieberman, Gerald J.; and Solomon, Herbert. Multi-level continuous sampling plans. *Ann. Math. Statist.* 26 (1955), 686-704.

The following generalization of Dodge's sampling inspection plan for continuous production [same *Ann.* 14 (1943), 264-279; *MR* 5, 130] is considered. Inspection can be carried out at any of the sampling rates $f_0=1$ (100% inspection), f_1, \dots, f_k , where $1/f_1, \dots, 1/f_k$ are integers. Inspection goes on at rate f_j ($j=0, 1, \dots, k$) until either a succession of i_j non-defective units is obtained or a defective unit is found. In the former case one proceeds to rate f_{j+1} ($f_{k+1}=f_k$), in the latter case to rate f_{j-1} ($f_{-1}=f_0$). All defective items found during inspection are assumed to be corrected or replaced with non-defective items. Assuming a statistically controlled production process and interpreting the sampling process as a Markov chain the authors prove that the AOQ (average outgoing quality=fraction defective after inspection) does not depend on the choice of initial sampling rate. They determine the AOQ explicitly and, for the particular case where $f_j=f$ and $i_j=i$, $\max(\text{AOQ})$ with respect to fraction defective before inspection, for $k=1, \infty$ and 2. Two criteria for the choice of f, i and k in this particular plan, viz. minimum average fraction inspected and local stability, are discussed. The latter is defined as maintained if the probability is $\leq \alpha$ that the number of defective items remaining after inspection in a sequence of N items will exceed NA , where N is large, α is a prescribed risk and A is the desired $\max(\text{AOQ})$.

D. M. Sandelius (Göteborg).

Lukaszewicz, J.; and Steinhaus, H. On measuring by comparison. *Zastos. Mat.* 2 (1955), 225-231. (Polish. Russian and English summaries)

Thickness being measured with n gauges $d_1 < \dots < d_n$, every s found to satisfy $d_j < s < d_{j+1}$ is taken to be O_j where O_0, \dots, O_n are fixed numbers satisfying $d_j < O_j < d_{j+1}$. Given the d.f. of s , the mean absolute (or squared) error is minimised with respect to O_0, \dots, O_n in terms of d_1, \dots, d_n , or vice versa, or with respect to both sets simultaneously. The numerical solution of the last problem for a normal d.f. and 15 gauges is given. The minimum number of trials is also discussed. *S. K. Zaremba.*

Kallianpur, G.; and Rao, C. Radhakrishna. On Fisher's lower bound to asymptotic variance of a consistent estimate. *Sankhyā* 15 (1955), 331-342; corrigenda 16 (1955), 206.

The authors seek to place adequate conditions on the distributions considered, and the corresponding classes of estimators for the unknown parameters of the distributions, so that estimators of super-efficiency are not

possible. They contend that R. A. Fisher has been misunderstood since they insist that his definition of consistency requires more than just that an estimator converge in probability to the corresponding parameter. [The reviewer questions their interpretation since he did not find it completely precise or clear.] The authors then consider two cases: 1) a multinormal distribution, 2) a continuous distribution. In each case they give sufficient conditions to insure that there exists a lower bound to the asymptotic variance of a consistent estimator. The estimators must be consistent according to their use of the term. *M. Muller (Ithaca, N.Y.).*

Smirnov, N. V. On the statistical estimation of transition probabilities in Markov chains. *Vestnik Leningrad. Univ.* 10 (1955), no. 11, 47-48. (Russian)

Let $\|p_{ij}\|$ be a stochastic matrix of order $s+1$ with strictly positive elements, and let (p_i) be the corresponding set of stationary absolute probabilities, with $p_i p_{ij} = p_j p_{ji}$. Let m_{ij} be the number of transitions from i to j in n trials. Then the probability of a specified matrix $\|m_{ij}\|$ is calculated, and this leads to a χ^2 test with s^2+s degrees of freedom that a specified stochastic matrix be the true one. *J. L. Doob (Urbana, Ill.).*

Romanovskii, V. I. Evaluation of a plan of statistical quality testing. *Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh.* 15 (1955), 11-17. (Russian)

The author proposes a method for the evaluation of a plan for acceptance sampling and also for the control of a production process. He defines a risk function and bases his decisions on A. Wald's minimax principle.

E. Lukacs (Washington, D.C.).

Romanovskii, V. I. On a statistical criterion of D. I. Mendelev. *Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh.* 15 (1955), 31-40. (Russian)

The author discusses a test for the homogeneity of a series of measurements which was suggested by a criterion proposed by D. E. Mendelev. Suppose that $3m$ measurements are taken; they are then randomly divided into 3 groups, each containing m measurements. Let $\bar{x}_1 \leq \bar{x}_2 \leq \bar{x}_3$ be the identically distributed means of the three groups. The author determines the joint distribution of the random variables $u = [(\bar{x}_2 - \bar{x}_1)\sqrt{m}]/\sigma$ and $v = [(\bar{x}_3 - \bar{x}_2)\sqrt{m}]/\sigma$ under the assumption that the observations are normally distributed with standard deviation σ . He then uses this result to find confidence intervals for the unknown standard deviation σ . A short table of the joint distribution of u and v and of its marginal distribution is also given. D. I. Mendelev's original proposal is somewhat more complicated: The observations are first arranged in increasing order and the three groups selected are the smallest, the central and the largest third of the ordered observations. Under these conditions the means of the groups are no longer identically distributed so that the determination of the joint distribution of u and v becomes much more complicated. The present author derives this distribution for a sample of size 6 and computes a brief table for it. *E. Lukacs (Washington, D.C.).*

Bhattacharya, P. K. Joint test for the mean and variance of a normal population. *Calcutta Statist. Assoc. Bull.* 6 (1955), 73-90.

Suppose that n independent observations are taken from a normal population with mean m and variance σ^2 . It is desired to test the hypothesis $H_0: (m=m_0, \sigma^2=\sigma_0^2)$,

where rejection must lead to either H_{01} : ($m=m_0$), H_{02} : ($\sigma^2=\sigma_0^2$), or to the rejection of both H_{01} and H_{02} . Three procedures are discussed. The first, T_1 , is the one which is probably used very often except that it is difficult to evaluate the constants of the test to obtain desired probabilities of error under H_0 . The second and third are easier to work with and tables for their application are included. The author compares these tests and concludes that T_1 is superior to T_2 and T_3 . T_2 is superior to T_3 when small deviations in the mean have to be detected and T_3 is superior to T_2 for detecting deviations mainly in the variance. In large samples all tests tend to become equivalent. *H. Chernoff* (Stanford University, Calif.).

Ura, Shoji. On the power function of Welch's test procedure in the two sample problems. Rep. Statist. Appl. Res. Un. Jap. Sci. Engrs. 4 (1955), 1-13.

The power of Welch's test for the difference between the means of two independent normal samples [Biometrika 34 (1947), 28-35; MR 8, 394], using the approximation based on the t -distribution [ibid. 36 (1949), 293-296; MR 11, 527] is calculated when the two population variances are equal (so that the statistic has the non-central t -distribution). For small degrees of freedom exact computations are performed: otherwise asymptotic results are obtained using Hartley's formulae [ibid. 33 (1944), 173-180; MR 6, 10]. The author's conclusion is that the Welch test and the t -test have nearly equal power unless the degrees of freedom are very unequal, and that therefore Welch's procedure is the better unless we are sure that the variances are equal. *D. V. Lindley.*

Chandra Sekar, C.; Agarwala, S. P.; and Chakraborty, P. N. On the power function of a test of significance for the difference between two proportions. Sankhyā 15 (1955), 381-390.

The authors determine the power function for the hypothesis $H_0: p_1=p_2=p$ when p_1 and p_2 have a common unspecified binomial distribution with equal sample sizes $n_1=n_2=n$. The size of the best critical region (which is carefully defined) when two-sided is less than or equal to α ; for one-sided alternatives it is less than or equal to $\alpha/2$. The power is tabled for essentially all combinations of p_1 and $p_2=.1(.1).9$ and $n=5(5)20(10)50, 100, 200, \alpha \leq .05$, for the two-sided test; for the one-sided test the power function is given for the same values of n, p_1 , and p_2 but $\alpha \leq .025$. For samples of $n \geq 50$ the power function changes so rapidly that a grid for $p < .1$ would be desirable. Examples illustrate the theory. *L. A. Aroian.*

Sakaguchi, Minoru. Notes on statistical applications of information theory. II. Rep. Statist. Appl. Res. Un. Jap. Sci. Engrs. 4 (1955), 57-68.

[For part I see same Rep. 1 (1952), no. 4, 27-31; MR 14, 996.] The author first defines a particular communication system with noise, and derives an inequality relating Shannon's channel capacity [C. E. Shannon and W. Weaver, The mathematical theory of communication, Univ. of Illinois Press, 1949; MR 11, 258] for the particular system defined and the measure of divergence between two probability measures discussed by Kullback and Leibler [Ann. Math. Statist. 22 (1951), 79-86; MR 12, 623]. He next considers the problem of estimating the transmitted symbol on the basis of the received symbol. (This problem is the test of two simple hypotheses with fixed sample size.) His theorem, a restatement of results of Mourier [C. R. Acad. Sci. Paris 223 (1946), 712-714;

Trabajos Estadíst. 2 (1951), 233-260; MR 8, 162; 14, 191], aims to give a statistical assertion to the statement that "divergence measures how difficult it is to discriminate between two probability measures with the best test". The author shows that the test described by Mourier in the C.R. note cited above is asymptotically minimax and that the asymptotically minimax risk is given in terms of the normal distribution. His last theorem generalizes the preceding result to the sequential case.

The author defines, for the sequential test T , the risk functions $r(0, T, c) = \alpha + cE_0(n|T)$ and $r(1, T, c) = \beta + cE_1(n|T)$. The test T_e^0 is given by: Take N_e observations and accept μ_0 or μ_1 according as

$$N_e^{-1} \sum_{i=1}^{N_e} \log \frac{f_1(x_i)}{f_0(x_i)} \leq \text{or} > \frac{\sigma_0 I_1 - \sigma_1 I_0}{\sigma_0 + \sigma_1},$$

where N_e is the smallest integer \geq

$$\left(\frac{\sigma_0 + \sigma_1}{I_0 + I_1} \right)^2 U^3 \left(- \left(\frac{1}{I_0 + I_1} - 2 \left(\frac{\sigma_0 + \sigma_1}{I_0 + I_1} \right)^2 \right) c \log c \right),$$

$$S = \int_{U(S)}^{\infty} (2\pi)^{-1/2} e^{-t^2/2} dt,$$

$$d\mu_0 = f_0(x) d\lambda, \quad d\mu_1 = f_1(x) d\lambda, \quad I_0 = \int f_0(x) \log \frac{f_0(x)}{f_1(x)} d\lambda,$$

$$I_1 = \int f_1(x) \log \frac{f_1(x)}{f_0(x)} d\lambda,$$

$$\sigma_i^2 = \int \left(\log \frac{f_1(x)}{f_0(x)} \right)^2 d\mu_i - I_i^2 \quad (i=0, 1).$$

Theorem: If $\sigma_0 + \sigma_1 \leq (\frac{1}{2}I_0 + \frac{1}{2}I_1)^{1/2}$, T_e^0 is an asymptotically minimax test for deciding between μ_0 and μ_1 , and its asymptotically minimax risk is

$$-2 \left(\frac{\sigma_0 + \sigma_1}{I_0 + I_1} \right)^2 c \log c.$$

As his last result the author derives for normal distribution functions the inequality

$$\phi(x) \log \frac{\phi(x)}{1-\phi(y)} + (1-\phi(x)) \log \frac{1-\phi(x)}{\phi(y)} \leq \frac{1}{2}(x+y)^2, \\ -\infty < x, y < \infty.$$

This inequality and special cases derived from it are related to results of Paulson [Ann. Math. Statist. 18 (1947), 447-450; MR 9, 152], Hammersley [J. Roy. Statist. Soc. Ser. B. 12 (1950), 192-240; MR 12, 725], Sampford [Ann. Math. Statist. 24 (1953), 130-132; MR 14, 995] and Tate [ibid. 24 (1953), 132-134; MR 14, 995].

S. Kullback (Washington, D.C.).

Kendall, M. G. Further contributions to the theory of paired comparisons. Biometrics 11 (1955), 43-62.

Expository paper on recent researches on paired comparisons. According to Wei [unpublished thesis, Cambridge, 1952], if A is the preference matrix, the powers of A give a score which corresponds to the "transitive preferences", i.e. the initial preferences recursively weighted by the other object scores. Since $A^n \lambda_1^{-n}$ tends to a limit, a more objective scoring may be obtained. In the second part some applications of incomplete block designs to paired comparisons are discussed.

M. P. Schützenberger (Paris).

Romanovskii, V. I. Sequential statistical control of the course of production. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. 15 (1955), 3-10. (Russian)

The author discusses a simple method for the statistical control of a production process. He assumes that the characteristic of the sample used in the plan is normally distributed. The proposed plan should however be called a continuous sampling plan rather than a sequential plan since samples of fixed size are taken at certain equidistant time points.
E. Lukacs (Washington, D.C.).

Romanovskii, V. I. The scheme of a machine and a measuring instrument. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. 15 (1955), 19-29. (Russian)

The author constructs several models for situations in which the observed random fluctuations are caused by two independent agents. We mention only the simplest case discussed by the author: The n measurements y_1, \dots, y_n are normally distributed about a mean x which is itself a random variable, the variance of the y_j is σ_1^2 . It is assumed that x, y_1, \dots, y_n are independently distributed and that x is normal with mean a and variance σ^2 . The quantities a, σ^2, σ_1^2 are unknown. Let $\bar{y} = (\sum y_j)/n$; a suitable t -statistic is derived for the investigation of the difference $\bar{y} - a$.
E. Lukacs (Washington, D.C.).

See also: Mitropol'skii, p. 702; Tintner, p. 760.

Theory of Games, Mathematical Economics

Zięba, A. Continuous games with perfect information. Bull. Acad. Polon. Sci. Cl. III. 3 (1955), 515-518.

The author considers the following game: a pursuer P is chasing an evader E along the real line. The pursuer's velocity $\dot{x}_P(t)$ must lie between γ and δ , the evader's $\dot{x}_E(t)$ between α and β , where $0 < \alpha < \beta < \gamma < \delta$. The evader has a head start, i.e. $x_P(0) < x_E(0)$. Under these conditions it follows that $x_P(t) = x_E(t)$ for exactly one instant $t = T$. Denote the common position of P and E at time T by x_T . A strategy for P consists of choosing a function f_P of x_P and x_E and determining his velocity by the rule $\dot{x}_P = f_P(x_P, x_E)$, where f_P is restricted to be of class C^2 . Similarly, E chooses a function f_E so that $\dot{x}_E = f_E(x_P, x_E)$, and of course $\alpha \leq f_E \leq \beta$ and $\gamma \leq f_P \leq \delta$. Finally, the real line is partitioned into two sets A_P and A_E and the game is a win for P or E according as $x_T \in A_P$ or $x_T \in A_E$. Theorem: If A_P and A_E are dense in the real numbers then the game is not strictly determined. The proof is sketchy.
D. Gale (Providence, R.I.).

Kuhn, H. W. The Hungarian method for the assignment problem. Naval Res. Logist. Quart. 2 (1955), 83-97.

Given n individuals and n jobs let a_{ij} be the rating of the i th individual in the j th job. The assignment problem then consists in determining a one-to-one correspondence between individuals and jobs which will minimize the sum of the ratings. This problem can be reformulated as a problem in linear programming and solved by standard methods. Here, instead, the author develops another algorithm for the case where the ratings a_{ij} are positive integers, based on an ingenious use of the "dual" problem, namely: to find numbers u_1, \dots, u_n and v_1, \dots, v_n such that $u_i + v_j \geq a_{ij}$ and $\sum u_i + \sum v_j$ is a minimum. It is known that this minimum is equal to the desired maximum rating. By means of the dual problem the author reduces the general problem to the special case where the

ratings a_{ij} are either 1 or 0 and develops a simple iterative procedure for solving this special case. The main feature of the method is that it is essentially combinatorial, the only operations involved consisting of adding or subtracting 1. In particular, it is never necessary to solve any simultaneous equations. (The proof of Theorem 1 contains an error but the reader should find little difficulty in patching it up.)
D. Gale (Providence, R.I.).

Dantzig, George B. Linear programming under uncertainty. Management Sci. 1, (1955) 197-206.

Among the class of problems which come under the general heading of "linear programming under uncertainty", the author selects for consideration a subclass which may be typified by the following: Let $K(\theta)$ be a convex polyhedron in $(m+n)$ -space (whose points are denoted by the vector couple (x, y)), such that the projection of $K(\theta)$ on m -space is a fixed set Ω , independent of θ . Let $\phi(x, y)$ be a convex function. Then

$$\phi(x) = \text{Exp} \left[\inf_{y | (x, y) \in K(\theta)} \phi(x, y) \right]$$

is a convex function of x . Extensions of the methods of linear programming may then be used (if the size of the problem is not too large) to compute the best choice of x in Ω . A variety of applications of these ideas are discussed. They have in common that a decision must be made at one stage to meet demands occurring at a later stage. These demands are now known only probabilistically, but will be known exactly at the time that second-stage decisions have to be made.
A. J. Hoffman.

*** Danskin, J. M. Linear programming in the face of uncertainty: example of a failure.** Proceedings of the Second Symposium in Linear Programming, Washington, D.C., 1955, pp. 39-53. National Bureau of Standards, Washington, D.C., 1955.

After entering into considerable detail on a proposed use of linear programming to calculate how to load weapons on an aircraft carrier in a tactical force, the author abandoned his model without computing any answers. His principal reasons were: the necessity for integral solutions in circumstances where "rounding" would lead to obvious nonsense, and uncertainty about the objective function. If the model were not so large that the computations were onerous, most linear programmers would probably feel the computations, in some form, should be undertaken. Because of the size, however, the issue is debatable. The reviewer believes that "failures" of this type are more likely to be instructive than trivial "successes".
A. J. Hoffman (Washington, D.C.).

*** Johnson, Selmer; and Dantzig, George. A production smoothing problem.** Proceedings of the Second Symposium in Linear Programming, Washington, D.C., 1955, pp. 151-176. National Bureau of Standards, Washington, D.C., 1955.

A single item is to be produced over a given number of time periods to satisfy known future requirements, in such a way that the sum of the costs of production, storage, and fluctuations in production rate are minimized. A rapid graphical method for solving this linear programming problem, involving only interactions and rotations of straight lines, is presented by the authors. Special results previously obtained on this problem by the reviewer and Jacobs [Management Sci. 1 (1954), 86-91; MR 17, 507] are easily deduced from the present computing scheme.
A. J. Hoffman (Washington, D.C.).

★ **Votaw, D. F., Jr.** *Programming under conditions of uncertainty.* Proceedings of the Second Symposium in Linear Programming, Washington, D.C., 1955, pp. 187-195. National Bureau of Standards, Washington, D.C., 1955.

★ **Tintner, G.** *Stochastic linear programming with applications to agricultural economics.* Proceedings of the Second Symposium in Linear Programming, Washington, D.C., 1955, pp. 197-228. National Bureau of Standards, Washington, D.C., 1955.

Assume that the coefficients of the matrix, bill of goods and objective function of a linear-programming problem are parameters only known probabilistically. If we restrict the space of these parameters so that, for any choice, the problem has a solution, then (in principle) the distribution of the optimum value of the objective function can be computed. The author's method for attacking the problem is to find the region in the parameter space at which each choice of basis column is both feasible and optimal, and to use approximations to determine the distribution of the solution vector. Examples involving one inequality in two nonnegative variables and two inequalities in two nonnegative variables are carried out in detail. The method does not seem to the reviewer capable of execution on problems of even modest size with the best computing equipment available or envisioned, but neither does any other method proposed to treat linear programming under uncertainty in this generality. *A. J. Hoffman* (Washington, D.C.).

★ **Radner, Roy.** *The linear team: an example of linear programming under uncertainty.* Proceedings of the Second Symposium in Linear Programming, Washington, D.C., 1955, pp. 381-396. National Bureau of Standards, Washington, D.C., 1955.

If each of the activity levels of x_j of a linear-programming problem is controlled by a separate decision-maker, knowing only the partial information y_j about the precise coefficients of the matrix, bill of goods and objective function (although the probability distribution of these coefficients is known to all), then the group of decision-makers are a "linear team". The author shows that the problem of finding the best decision rules is a linear programming problem in the space of decision functions and that the dual leads to a system of probability distributions of implicit prices. *A. J. Hoffman.*

★ **Thrall, R. M.** *Some results in non linear programming.* Proceedings of the Second Symposium in Linear Programming, Washington, D.C., 1955, pp. 471-493. National Bureau of Standards, Washington, D.C., 1955.

The author considers some problems in non-linear programming in which one seeks effective means of computing solutions whose existence has been previously established. Typical of the results is a formal prescription, in terms of partial derivatives, for computing the minimum of the sum of strictly convex differentiable functions of one variable. *A. J. Hoffman* (Washington, D.C.).

★ **Schell, Emil D.** *Distribution of a product by several properties.* Proceedings of the Second Symposium in Linear Programming, Washington, D.C., 1955, pp. 615-642. National Bureau of Standards, Washington, D.C., 1955.

While the one-commodity transportation problem has been studied extensively, very little work has been done

on the multi-commodity case. This paper represents an effort in that direction. Let n commodities be produced at l different origins, b_{ik} being the amount of the k th good produced at the i th origin, and let there be m destinations, where a_{jk} is the amount of the k th commodity demanded at the j th destination. Finally let c_{ij} be the capacity of the route from the i th origin to the j th destination. Problems of this sort lead naturally to questions on the solvability of systems of the form

$$(1) \sum_i x_{ijk} = a_{jk}, \sum_j x_{ijk} = b_{ik}, \sum_k x_{ijk} = a_{ik}, \text{ and } (2) x_{ijk} \geq 0.$$

Obvious necessary conditions for the system to have a solution are

$$(3) \sum_k b_{ik} = \sum_j c_{ij}, \sum_k a_{jk} = \sum_i c_{ij}, \sum_j a_{jk} = \sum_i b_{ik}.$$

$$(4) a_{jk} \geq 0, b_{ik} \geq 0 \text{ and } c_{ij} \geq 0.$$

These conditions are, however, not sufficient as is easily shown by examples. The author gives an additional necessary condition. Let $m_{ijk} = \min(a_{jk}, b_{ik}, c_{ij})$; then since x_{ijk} cannot exceed m_{ijk} , another necessary condition for solvability is

$$(5) \sum_i m_{ijk} \geq a_{jk}, \sum_j m_{ijk} \geq b_{ik}, \sum_k m_{ijk} \geq c_{ij}.$$

The author asserts that (3), (4) and (5) are also sufficient for solvability of (1) and (2). This assertion is unfortunately not correct and this invalidates much of the remainder of the paper which describes a method for solving these problems based on this false premise. The author also asserts that if any of the inequalities (5) are not equations then the solution if it exists is non-unique. This too is false. Finally the statement on basic feasible solutions on page 629 is also false as are the computational consequences which are derived from it.

The paper also contains brief discussion of similar problems in which, for instance, the double sums $\sum_{i,j} x_{ijk}$, $\sum_{i,k} x_{ijk}$ and $\sum_{j,k} x_{ijk}$ are prescribed.

D. Gale (Providence, R.I.).

Kelley, J. E., Jr. *A dynamic transportation model.* Naval Res. Logist. Quart. 2 (1955), 175-180.

The author discusses a transportation model originally due to Heller [Symposium on Linear Inequalities and Programming, Washington, D.C., June, 1951, pp. 164-171, Project Scoop rep. no. 10 (1952)]. Let A_i ($i=1, \dots, m$) be the number of loads available at the i th lifting port, B_j ($j=1, \dots, n$) number needed at the j th destination port, both over the whole year. Let a_i, b_j be the part of the availabilities and requirements in the first month. Empty travel cost c_{ij} and steaming time t_{ij} are known for all (i, j) . All figures are integers. The problem is to choose a feasible schedule x_{ij}, y_{ij} , where x_{ij} is the number of ships sent empty from i to j in the first month and y_{ij} the same during the rest of the year, so as to minimize total cost. The problem is given an explicit formulation, and it is then shown that it can be transformed in a transportation problem of the classical Hitchcock-Koopmans type, so that the simplex method will yield a solution in integers.

K. J. Arrow (Stanford, Calif.).

Ross, Ian C.; and Harary, Frank. *Identification of the liaison persons of an organization using the structure matrix.* Management Sci. 1 (1955), 251-258.

If the structure of an organization is represented by a linear graph, then the liaison persons are associated with the articulation points of this graph. In this paper a

condition is developed, in terms of distance along the graph, that a point W of the graph be an articulation point.

C. C. Torrance (Monterrey, Calif.).

Marschak, J. Elements for a theory of teams. *Management Sci.* 1 (1955), 127-137.

A team is a group of persons each of whom takes decisions about something different but who receive a common reward as the joint result of all those decisions. The concept of gross score is introduced as a function of: situation variables x_i ; decision variables; probability distributions of situations; decision rules based on each members' information about the x_i ; communication networks between members; interaction between decisions. The problem of maximizing net score = gross score minus cost of information is illustrated by a simple example.

C. C. Torrance (Monterrey, Calif.).

Richards, Paul I. Shock waves on the highway. *Operations Res.* 4 (1956), 42-51.

The flow of highway traffic is considered in the large, i.e. on a scale of distances such that one can define an effective density, D , of automobiles at any point x along the highway and at any time, t . The average velocity V of the cars at (x, t) is assumed to satisfy an empirical relation $V = a(b - D)$ in which a and b are constants depending upon the characteristics of the highway. This relation along with the equation describing of conservation automobiles

$$\frac{\partial D}{\partial t} + \frac{\partial (VD)}{\partial x} = 0$$

defines a non-linear wave equation for D . This simple first-order wave equation predicts the formation of "shock waves" analogous to those in the theory of fluids. Cars in a region of low density travel faster than those in a region of high density and so a discontinuity in D builds up as the fast moving cars overtake the slower ones. Several examples and simple geometrical constructions are given to illustrate the theory.

Most of the ideas contained in this paper were described somewhat earlier by Lighthill and Witham [*Proc. Roy. Soc. London. Ser. A* 229 (1955), 317-345; MR 17, 310]. The latter paper actually contains a more general and more detailed analysis but the two papers are sufficiently different in style as to supplement each other.

G. Newell (Providence, R.I.).

See also: Bellman, Glicksberg and Gross, p. 752; Markowitz, p. 789.

Mathematical Biology

Ceppellini, R.; Siniscalco, M.; and Smith, C. A. B. The estimation of gene frequencies in a random-mating population. *Ann. Human Genetics* 20 (1955), 97-115.

A general method is given of numerically evaluating gene frequencies in a large population in equilibrium under panmixia. The procedure for finding the standard errors of the estimates is indicated. Finally, it is shown that the method is equivalent to maximum likelihood.

Y. Komatu (Tokyo).

Trucco, Ernesto. A note on Rashevsky's theorem about point-bases in topological biology. *Bull. Math. Biophys.* 18 (1956), 65-85.

Rashevsky, N. The geometrization of biology. *Bull. Math. Biophys.* 18 (1956), 31-56.

Kostitzin, Vladimir. Sur le développement des populations bactériennes. *C. R. Acad. Sci. Paris* 242 (1956), 611-612.

de Donder, Th. Le calcul des variations introduit dans la théorie des espèces et des variétés. XIV. *Acad. Roy. Belg. Bull. Cl. Sci.* (5) 41 (1955), 1104-1105.

TOPOLOGICAL ALGEBRAIC STRUCTURES

Topological Groups

Leptin, Horst. Ein Darstellungssatz für kompakte, total unzusammenhängende Gruppen. *Arch. Math.* 6 (1955), 371-373.

It is proved that every totally disconnected compact topological group is the Galois group of some normal extension of a field. In order to do this, the group under consideration is imbedded into a product of a family of finite groups, each of which may be assumed to be the symmetric group of some degree.

C. Chevalley (Paris).

Ganea, Tudor. Groupes topologiques sans centre. *Rev. Univ. "C. I. Parhon" Politehn. București. Ser. Ști. Nat.* 2 (1953), no. 3, 37-38. (Romanian. Russian and French summaries)

Let ϕ be an open local homomorphism $G \rightarrow H$ of topological groups, where G is locally connected and H is connected. It is shown in an elementary manner that if the center of H is trivial, ϕ can be extended in a unique way to a homomorphism $G \rightarrow H$.

P. A. Smith.

Matsushita, Shin-ichi. Sur quelques types des théorèmes de dualité dans les groupes topologiques. I, II. *Proc. Japan Acad.* 30 (1954), 849-854, 957-962.

Denote by G a topological group, by $A(G)$ the commu-

tative C^* -algebra of almost periodic (a.p.) functions on G with the uniform topology, by $\Phi^0(A(G))$ the set of all multiplicative linear functionals on $A(G)$ with the topology of simple convergence, by \tilde{G} the natural image of G in $\Phi^0(A(G))$, by $\mathcal{E}(G)$ the group of all bounded regular linear operators S on $A(G)$ such that $(Sf)^* = S^*/f$, $S(fg) = S(f)S(g)$, $S(1) = 1$, by \mathcal{E}_G (resp. \mathcal{E}_G^0) the subgroup of all operators ${}_a S$ (resp. S_a) defined as follows: $({}_a S f)(x) = f(a^{-1}x)$ (resp. $(S_a f)(x) = f(xa)$), where $a \in G$, and by \mathcal{E}_G^0 (resp. ${}^0 \mathcal{E}_G$) the centralizer of \mathcal{E}_G (resp. \mathcal{E}_G^0) in $\mathcal{E}(G)$. Let G be maximally a.p. It is shown that in two different ways $\Phi^0(A(G))$ can be made a group which is canonically isomorphic with \mathcal{E}_G^0 or ${}^0 \mathcal{E}_G$ respectively; these isomorphisms induce on G the natural isomorphisms $\mathcal{E}_G \cong G \cong {}^0 \mathcal{E}_G$. Moreover, $\mathcal{E}(G)$ can be given a suitable topology (the "weak" topology) which, restricted to \mathcal{E}_G^0 and ${}^0 \mathcal{E}_G$, is compatible with that of $\Phi^0(A(G))$, and which makes \mathcal{E}_G^0 and ${}^0 \mathcal{E}_G$ compact topological groups. This gives a new formulation on the Tannaka duality theorem [*Tôhoku Math. J.* 45 (1938), 1-12]. A generalization is given for groups which are not maximally almost periodic.

Now let G be locally compact abelian and let \hat{G} denote the Pontrjagin duality. The author defines on $\mathcal{E}_G^0 = {}^0 \mathcal{E}_G$ and on \tilde{G} new topologies such that he can prove the following topological isomorphisms: $G \cong {}^0 \mathcal{E}_G = \mathcal{E}_G \cong \tilde{G}$ and

$G \cong G \cong \hat{G}$; this provides a new proof for the Pontrjagin duality theorem. Other groups, defined like $\Phi^0(A(G))$ but starting from another topology on $A(G)$, are shown to be isomorphic with $\hat{G} \cong G$. A last theorem concerns the mean value of an a.p. function on the group G . *J. L. Tits.*

Jacoby, Robb. One-parameter transformation groups of the three-sphere. *Proc. Amer. Math. Soc.* 7 (1956), 131-142.

Let K be the additive group of reals mod 1 and S the unit three-sphere in the Euclidean four-space E^4 . For any relatively prime positive integers k, l and for any real number t , let $f_{k,l}(t)$ denote the rotation of E^4 around the origin, composed of the rotation of (x_1, x_2) -plane with angle $2\pi kt$ and the rotation of (x_3, x_4) -plane with angle $2\pi lt$. The mapping $K \times S \rightarrow S$ defined by $i \times x \rightarrow f_{k,l}(i)x$ ($x \in S, i = t \bmod 1$) then gives a transformation group for the pair (K, S) .

Now, let there be given any transformation group for the pair (K, S) and let $i \times x \rightarrow f(i)x$ be the mapping defining its structure. In the present paper, the author proves that if the transformation group is effective and has no fixed point, there exist integers k, l and a homeomorphism φ of S onto itself such that $\varphi f \varphi^{-1} = f_{k,l}$.

The idea of the proof is roughly as follows: the intersection of S and (x_1, x_2) -plane (or (x_3, x_4) -plane) is an orbit of the $f_{k,l}$ -group with some exceptional property (so far as $k > 1$ or $l > 1$). The author, therefore, studies the corresponding exceptional orbits of the given group defined by f , finds local cross-sections around those orbits, proves that there are at most two such exceptional orbits, and then defines a homeomorphism φ as described above.

K. Iwasawa (Cambridge, Mass.).

See also: Nussbaum, p. 771; Mostert, p. 771; Heller, p. 773.

Lie Groups, Lie Algebras

Dynkin, E. B.; and Oniščik, A. L. Compact global Lie groups. *Uspehi Mat. Nauk (N.S.)* 10 (1955), no. 4(66), 3-74. (Russian)

The first purpose of the paper is to develop the theory of the diagram of a compact Lie group [in the sense of Stiefel, *Comment. Math. Helv.* 14 (1942), 350-380, pp. 375-379; MR 4, 134], but using now exclusively the Cartan-Weyl theory of root forms as the basic tool. Using the diagram the centres of the simply connected compact simple Lie groups are determined. Further, the kernels of these groups under a unitary representation are determined. The result is stated in terms of the dominant weights of the irreducible components. Among the other results we mention a criterion for an irreducible linear representation of a compact Lie group to be equivalent to a symplectic or orthogonal representation.

W. T. van Est (Utrecht).

Dobrescu, Andrei. La classification des groupes de Lie à quatre paramètres, à vecteur de structure nul. *Com. Acad. R. P. Române* 2 (1952), 665-668. (Romanian. Russian and French summaries)

The author classifies all real 4-dimensional Lie algebras with $c_{ik} = c_{ki} = 0$, where c_{ik} denotes the structure constants; this completes a previous result of his concerning the case where the vector c_k does not vanish [*Acad. R. P. Române Bul. Şti. Ser. Mat. Fiz. Chim.* 2 (1950), 137-146;

MR 14, 1062]. In a later paper [*Acad. R. P. Române. Stud. Cerc. Mat.* 4 (1953), 395-436; MR 16, 567] both results have been re-expounded in greater detail.

J. L. Tits (Brussels).

Murakami, Haruo. A note on exponential mapping. *Portugal. Math.* 14 (1955), 15-19.

An unnecessarily complicated proof of the fact that a complex invertible matrix is the exponential of some matrix. *C. Chevalley (Paris).*

Laugwitz, Detlef. Über unendliche kontinuierliche Gruppen. I. Grundlagen der Theorie; Untergruppen. *Math. Ann.* 130 (1955), 337-350.

This is an essay in the construction of a general theory of infinite continuous groups with the use of vectors from a Banach space to form a local coordinate system. The author makes use of the tensor calculus developed by him in a previous paper [*Math. Z.* 61 (1954), 100-118; MR 16, 512]. A topological group G is called B -locally differentiable (or analytic) if the following conditions are fulfilled: Some neighborhood of the neutral element e in G is homeomorphic with a neighborhood of the zero point 0 in the Banach space B , with e mapping into 0, and this homeomorphism carries the group product into a function on $B \times B$ which is at least twice continuously differentiable (or analytic), using Fréchet differentials. One may also consider B -locally differentiable group 'germs' or local-groups. The author discusses right and left parallelism, derives two linear connections associated with parallel displacements, and proves a uniqueness and existence theorem concerning a one-parameter group which furnishes the geodesic lines for each of these connections. It is possible to establish the coordinate system in such a way that this one-parameter group has a certain very simple form; then the coordinate system is called canonical. In this way it is proved that every one-parameter subgroup is differentiable in each differentiable coordinate system. After using existence theorems of Kerner [*Prace Mat.-Fiz.* 40 (1932), 47-67], the argument is much as in Pontrjagin's "Topological groups" [Princeton, 1939, Th. 48; MR 1, 44]. There are some further differentiability theorems on subgroups. In particular, application is made to the group G of regular elements of a real Banach algebra B . This group is an analytic B -group. A subgroup H of G is a locally analytic B' -group (where B' is a certain subspace of B) if (a) H is locally compact or (b) H is the center of G . The (a) case, for complex Banach algebras, was proved in a completely different manner by Yosida [*Jap. J. Math.* 13, 7-26 (1936), 459-472 (1937)].

A. E. Taylor (Los Angeles, Calif.).

Cartier, Pierre. Dualité de Tannaka des groupes et des algèbres de Lie. *C. R. Acad. Sci. Paris* 242 (1956), 322-325.

Let G be a group. A set R of finite-dimensional linear representations of G over a field K is called a ring of representations if it is stable under the operations of forming the dual representation, subrepresentations, factor representations, the direct sum of representations, and the tensor product of representations. A representation of a ring R of representations of G is a function f associating with each representation $\rho \in R$ a non-singular linear transformation $f(\rho)$ of the corresponding representation space, in such a way that the natural compatibility relations hold with respect to the "ring operations" in R . These representations of R constitute a group, denoted

G_R , in the natural way. If g is any element of G , and if $f_g(e) = e(g)$, then the map $g \rightarrow f_g$ is a homomorphism of G into G_R . One says that the Tannaka duality holds for R if this homomorphism is an isomorphism of G onto G_R . Exactly analogous definitions are made for Lie algebras \mathfrak{G} and their representations.

The following theorems are announced: (1) In order that R be the set of all (up to equivalence) rational representations for a structure of a linear algebraic group on G it is necessary and sufficient that R be "generated" by a finite set of representations, and that the Tannaka duality hold for R . (2) If K is of characteristic 0, then the ring of all rational representations of an algebraic group G is generated by any faithful rational representation of G . (1') and (2') Let \mathfrak{G} be a Lie algebra over a field of characteristic 0. In order that the Tannaka duality hold for a ring \mathfrak{R} of representations of \mathfrak{G} it is necessary and sufficient that \mathfrak{G} be the Lie algebra of an algebraic group G and that \mathfrak{R} be the ring of all representations of \mathfrak{G} that are derived from rational representations of G . In that case, if ρ is any faithful representation of \mathfrak{G} that belongs to \mathfrak{R} , there is a finite set ρ_1, \dots, ρ_n in \mathfrak{R} such that every member of \mathfrak{R} is a factor representation of a representation contained in a direct sum of tensor products of the form $(\otimes \rho_i) \otimes \rho_j$. (3) If, furthermore, K is algebraically closed then the Tannaka duality holds if and only if the following two conditions are satisfied: (a) If \mathfrak{K} is the intersection of the kernels of all simple representations belonging to \mathfrak{R} then every member of \mathfrak{R} that annihilates \mathfrak{K} is semisimple; (b) \mathfrak{K} is contained in the radical \mathfrak{R} of \mathfrak{G} , and the rank over the integers of the set of weights of \mathfrak{R} for the representations belonging to \mathfrak{R} is equal to the dimension of $\mathfrak{R}/\mathfrak{K}$.

Theorems (2') and (3) include the main results of Harish-Chandra [Ann. of Math. (2) 51 (1950), 299-330; MR 11, 492]. The following is also stated to be a consequence of the above results: if \mathfrak{G} is semisimple, then among the algebraic groups G with Lie algebra \mathfrak{G} there is one, G_0 , such that all others are factor groups of G_0 by a subgroup of the finite center of G_0 . If K is algebraically closed and if r is the rank of \mathfrak{G} then the center of G_0 is generated by r elements m_1, \dots, m_r satisfying the relations $\sum_i a_i m_i = 0$, where the a_i are the Cartan integers of \mathfrak{G} .

G. P. Hochschild (Berkeley, Calif.).

Dieudonné, Jean. Sur la notion de variables canoniques. An. Acad. Brasil. Ci. 27 (1955), 251-258.

Correction and complement to Comment. Math. Helv. 28 (1954), 87-118 [MR 16, 12], correcting the proof for the existence of canonical variables, and proving that a formal Lie group whose group law is canonical is completely determined by its hyperalgebra and that to every "typical" subalgebra of the hyperalgebra there actually corresponds a "typical" subgroup. G. P. Hochschild.

Sugiura, Mitsuo. On a certain property of Lie algebras. Sci. Papers Coll. Gen. Ed. Univ. Tokyo 5 (1955), 1-12.

The author calls a Lie algebra \mathfrak{g} an (A) -algebra if the condition $[X, [X, Y]] = 0$ (with X, Y in \mathfrak{g}) implies $[X, Y] = 0$. These algebras have been studied by I. M. Singer [Ann. of Math. (2) 56 (1952), 242-247; MR 14, 135] who classified them in the case where the basic field is that of real numbers. The author shows that a Lie algebra \mathfrak{g} over a field of characteristic 0 is an (A) -algebra if and only if it is reductive and its derived algebra is composed of semi-simple elements. The analytic groups whose Lie algebras are (A) -algebras are the maximally almost

periodic groups, which may also be characterized as being the products of vector groups and compact analytic groups, or as having a neighbourhood of the unit element which is invariant under all inner automorphisms; it is shown that the locally compact maximally almost periodic groups are the projective limits of analytic groups with the same property. C. Chevalley.

See also: Lazard, p. 710; Boerner, p. 710; Yokota, p. 774.

Topological Vector Spaces

Grothendieck, Alexandre. Produits tensoriels topologiques et espaces nucléaires. Mem. Amer. Math. Soc. no. 16 (1955), 140 pp.

Le grand nombre de résultats importants contenus dans ce mémoire (Thèse), rend difficile d'en mettre en évidence les lignes essentielles, même si l'on a recours au résumé des résultats publié auparavant [Ann. Inst. Fourier, Grenoble 4 (1952), 73-112; MR 15, 879, 1140]. L'origine concrète de ces recherches se trouve dans le dessein de créer une catégorie spéciale d'espaces localement convexes, formant le cadre naturel de théories comme celles des distributions et des fonctionnelles analytiques. L'auteur est ainsi arrivé à la notion d'espace nucléaire, qui généralise la propriété exprimée par le "théorème des noyaux" (voir ci-après l'analyse des exposés du Séminaire Schwartz), mais dont l'étude exige une technique très approfondie, concernant la théorie générale des produits tensoriels topologiques.

Le chap. I est consacré à une mise au point de cette théorie. On appelle produit tensoriel projectif de deux espaces localement convexes E, F le produit tensoriel algébrique $E \otimes F$, muni de la topologie, T_n , la plus fine, qui rende continue l'application $(x, y) \rightarrow x \otimes y$ de $E \times F$ dans $E \otimes F$. Un système fondamental de voisinages de 0 pour T_n est formé des enveloppes convexes cercles des produits $U \otimes V$ de deux voisinages U, V de 0 resp. dans E et F . Le complété de cet espace est noté $E \hat{\otimes} F$. On établit plusieurs propriétés de permanence de l'opération $(E, F) \rightarrow E \hat{\otimes} F$, par rapport au produit vectoriel topologique, à la somme directe, etc. Si E, F sont du type (\mathcal{F}) (resp. (\mathcal{DF})) il en est de même de $E \hat{\otimes} F$. Si E, F sont normables ou quasinormables ou de Schwartz, il en est de même de $E \hat{\otimes} F$. [La désignation "produit tensoriel projectif" tient au fait que ce produit se conserve dans le passage à limite projective. Si l'on remplace dans la définition précédente "continue" par "séparément continue", on obtient la notion de "produit tensoriel inductif"]. Dans le cas de deux espaces (\mathcal{F}) , les éléments de $E \hat{\otimes} F$ sont les sommes des séries absolument convergentes $\sum \lambda_n (x_n \otimes y_n)$, où (x_n) [resp. (y_n)] est une suite bornée dans E [resp. dans F] et $\sum |\lambda_n| < +\infty$. Un exemple important: Soit $L_E^1(\mu)$ l'espace des applications μ -sommables d'un espace localement compact M (muni d'une mesure μ) dans un espace de Banach E , et soit $L^1(\mu)$ l'espace des fonctions scalaires μ -sommables dans M ; alors $L^1(\mu) \hat{\otimes} E$ est isomorphe à $L_E^1(\mu)$. Ce résultat se généralise au cas d'un espace localement convexe E , quelconque.

Une autre variante topologique est introduite. En identifiant chaque élément $\sum x_i \otimes y_i$ (somme finie) de $E \otimes F$ à la forme bilinéaire $\sum \langle x', x_i \rangle \langle y', y_i \rangle$ en x', y' sur $E' \times F'$, on peut munir $E \otimes F$ de la topologie, T_n , de la convergence uniforme sur les produits $H \times K$, où H (resp. K) est un ensemble équicontinu de E' (resp. F').

Le complété de $E \otimes F$ pour T_ϵ est noté $E \hat{\otimes} F$. [Alors, si E, F sont complets, $E \hat{\otimes} F$ s'identifie à un sous-espace de l'espace $\mathcal{B}_\epsilon(E', F')$ des formes bilinéaires séparément faiblement continues sur $E' \times F'$, muni de la topologie de la convergence biéquicontinue; celui-ci, à son tour, est isomorphe à l'espace d'applications linéaires continues $L_\epsilon(E', F)$, où τ indique la topologie de Mackey et ϵ la topologie de la convergence équicontinue]. Il existe une application linéaire continue (canonique) $E \hat{\otimes} F \rightarrow E \hat{\otimes} F$, mais il reste encore à décider si cette application est toujours biunivoque („problème de biunivocité”).

Le produit tensoriel algébrique de deux applications linéaires continues, u_1 et u_2 , resp. de E_1 dans F_1 et de E_2 dans F_2 , défini par $(u_1 \otimes u_2)(x_1 \otimes x_2) = u_1 x_1 \otimes u_2 x_2$, se prolonge en une application linéaire continue $u_1 \otimes u_2$ (resp. $u_1 \hat{\otimes} u_2$) de $E_1 \hat{\otimes} E_2$ dans $F_1 \hat{\otimes} F_2$ (resp. de $E_1 \hat{\otimes} E_2$ dans $F_1 \hat{\otimes} F_2$). L'auteur établit des propriétés de permanence de ces opérations, concernant les cas où u_1, u_2 sont des homomorphismes ou des isomorphismes topologiques, et il en fait quelques applications intéressantes.

Soient E, F deux espaces de Banach. Chaque élément $u = \sum x'_i \otimes y_i$ de $E' \otimes F$ définit l'application de rang fini $\bar{u}(x) = \sum \langle x'_i, x \rangle y_i$, de E dans F ; chaque élément $v = \lim u_n$ de $E' \hat{\otimes} F$, avec $u_n \in E' \otimes F$, définit l'application linéaire continue $\bar{v} = \lim \bar{u}_n$ de E dans F (convergence uniforme sur les bornés de E). On appelle nucléaires les opérateurs \bar{v} ainsi obtenus. Dans le cas général, une application ϕ de E dans F est dite nucléaire, si elle est de la forme $\phi = \alpha \beta \gamma$, où β est une application nucléaire entre deux espaces de Banach E_1, F_1 , et γ, α sont des applications linéaires continues, resp. de E dans E_1 et de F_1 dans F . Toute application nucléaire de E dans F est compacte et s'exprime par une série $\sum \lambda_n \langle x'_n, x \rangle y_n$, où (x'_n) est une suite équicontinue dans E' , (y_n) une suite extraite d'un disque compact de F et $\sum |\lambda_n| < +\infty$ (et réciproquement). Ces opérateurs appartiennent à la catégorie des opérateurs de Fredholm, définie par l'auteur dans sa théorie générale de Fredholm (éffleurée ici et développée ailleurs), avec l'emploi des produits tensoriels inductifs; ces deux catégories d'opérateurs coïncident, par exemple, dans le cas des espaces de Banach. Une autre catégorie comprenant les opérateurs nucléaires est celle des applications de E dans F (dites „intégrales”) définies par les formes linéaires continues sur $E \hat{\otimes} F$; parmi plusieurs définitions équivalentes, l'auteur en donne une représentation au moyen d'intégrales. Le produit de deux applications intégrales est nucléaire.

Le chapitre I se termine par un grand nombre de formulations équivalentes du „problème d'approximation”, dont voici une variante: Toute application linéaire continue de E dans F est-elle limite, pour la convergence précompacte, d'applications linéaires continues de rang fini? (E, F espaces localement convexes quelconques). Une autre variante est le „problème de biunivocité”, indiqué plus haut. La réponse est affirmative, si E (ou F) vérifie la „condition d'approximation”: l'application identique de E dans E est adhérente à $E' \hat{\otimes} E$ dans $L(E, E)$ muni de la topologie de la convergence précompacte. La plupart des espaces connus ont cette propriété, qui s'étend probablement à tous les espaces l.c.

Le chapitre II est consacré aux espaces nucléaires. [Nous adopterons ici l'ordre suivi par l'auteur dans son résumé des résultats.] Un espace localement convexe E est dit nucléaire, si, pour tout espace localement convexe F , l'application canonique $E \hat{\otimes} F \rightarrow E \hat{\otimes} F$ est un isomorphisme

me vectoriel topologique de $E \hat{\otimes} F$ sur $E \hat{\otimes} F$; ou, ce qui revient au même, si toute application linéaire continue de E dans un espace de Banach est nucléaire. S'il en est ainsi, et si E, F sont complets, on a $E \hat{\otimes} F \cong \mathcal{B}_\epsilon(E', F') \cong L_\epsilon(E', F)$. Les parties bornées d'un espace nucléaire E sont précompactes [si, en outre, E est quasi complet, il est du type (\mathcal{M}) , donc réflexif]. Pour qu'un espace E du type (\mathcal{S}) soit nucléaire, il faut et il suffit que toute suite sommable dans E soit absolument sommable. Tout espace quotient d'un espace nucléaire est nucléaire; le produit vectoriel topologique d'une famille quelconque d'espaces nucléaires est nucléaire, et de même pour la somme directe d'une famille dénombrable ou le produit tensoriel projectif complété d'un nombre fini d'espaces nucléaires. Si un espace nucléaire est du type (\mathcal{S}) ou (\mathcal{DS}) , son dual fort est nucléaire, et réciproquement. Les espaces (\mathcal{E}) , (\mathcal{E}') , (\mathcal{D}) , (\mathcal{D}') , (\mathcal{O}) , (\mathcal{O}') , (\mathcal{O}_M) , (\mathcal{O}_ϵ) de distributions, ainsi que l'espace H des fonctions holomorphes dans une variété analytique complexe, sont nucléaires. Après quelques propriétés de relèvement, l'auteur démontre que, si E, F sont tous deux du type (\mathcal{S}) [resp. (\mathcal{DS})] et E nucléaire, on a $(E \hat{\otimes} F)' \cong E' \hat{\otimes} F'$, par rapport aux topologies fortes; le cas où E est du type (\mathcal{S}) et F du type (\mathcal{DS}) conduit en général à des situations moins régulières.

On appelle opérateurs de Fredholm de puissance p -ème sommable les applications de E dans F exprimables au moyen de séries $\sum \lambda_n x'_n \otimes y_n$ (par rapport à la topologie du produit tensoriel inductif de E' par F), avec $(\lambda_n) \in \mathbb{R}$, $0 < p \leq 1$, (x'_n) [resp. (y_n)] suite extraite d'un compact de E' [resp. F]. Si $p=1$, on retrouve les opérateurs de Fredholm, déjà cités; on appelle ordre d'un tel opérateur u la borne inférieure des p pour lesquels u est de puissance p -ème sommable. Toute application u de Fredholm de E dans F admet un „détérminant de Fredholm”, fonction entière de la variable complexe, dont les zéros sont les inverses des valeurs propres de u et dont l'ordre est lié à l'ordre de u par une certaine relation. Tout opérateur borné (c.a.d. transformant un voisinage convenable de 0 en un borné) dans un espace nucléaire est un opérateur de Fredholm d'ordre 0, dont les valeurs propres forment une suite à décroissance rapide.

Beaucoup d'autres résultats sont établis, dont il ne serait pas facile de donner ici un aperçu.

J. Sebastião e Silva (Lisbonne).

★ Séminaire Schwartz de la Faculté des Sciences de Paris, 1953/1954. Produits tensoriels topologiques d'espaces vectoriels topologiques. Espaces vectoriels topologiques nucléaires. Applications. Secrétariat mathématique, 11 rue Pierre Curie, Paris, 1954. iii+144 pp. (polycopiés).

Deux tiers des exposés de ce séminaire sont consacrés à l'étude de la Thèse de Grothendieck [voir l'analyse ci-dessus]. Tout le reste, exception faite d'une partie de l'exposé 24, est le résumé d'un travail inédit de L. Schwartz. La technique puissante des produits tensoriels topologiques est ici utilisée pour l'édification d'une théorie complète et générale de l'intégration, dans le domaine des distributions.

On considère d'abord l'intégration d'une fonction vectorielle différentiable par rapport à une distribution. On désigne par H^m un espace de fonctions numériques m -fois continuellement différentiables sur \mathbb{R}^n ($\mathcal{D}^m \subset H^m \subset \mathcal{E}^m$), vérifiant certaines conditions de croissance par rapport à un ensemble Γ de fonctions; on munit H^m d'une topologie convenable, telle que les immersions $\mathcal{D}^m \rightarrow$

$H^m \rightarrow \mathcal{E}^m$ soient continues. [Exemples: D^m ($\Gamma = \mathcal{E}^m$), S ($\Gamma =$ ensemble des polynômes, $m = \infty$), etc.] Soit E un espace localement convexe complet; $\tilde{H}^m(E)$ est l'espace des fonctions φ , m -fois continuellement différentiables, à valeurs dans E et scalairement dans H^m [c.a.d., telles que $\varphi' \in H^m$ pour tout $\varphi \in E$]. On a $\tilde{H}^m(E) \cong H^m \otimes E$. [$\tilde{S}(E) = S(E)$; mais en général $\tilde{D}(E) \neq D(E)$]. L'intégrale d'une fonction $\varphi \in \tilde{H}^m(E)$ par rapport à une distribution $T \in H^m$ est définie par $\int T_x \varphi(x) dx = T \otimes I_E(\varphi)$ (où I_E est l'application identique $E \rightarrow E$) et notée $T(\varphi)$. On a $T(\varphi) \in E$ et $T(v\varphi) = v(T\varphi)$ pour tout opérateur $v \in L(E, F)$, F étant complet [ce qui caractérise $T(\varphi)$, lorsque $F = C$]. Il est alors possible, p.ex., de donner une définition intégrale de convolution et de la transformation de Fourier $\mathcal{F}: S' \rightarrow S'$. La fonction $\delta(y-x)$ de x appartient toujours à $\tilde{H}^m(H_c^m)$, d'où ce résultat remarquable: $\tilde{H}^m(E) \cong L_c(H_c^m, E)$ (c indique la convergence compacte et e la convergence équicontinue).

Un autre point de vue est introduit. On considère l'espace \mathcal{D}'_L des distributions „sommables”, dual fort de l'espace \mathcal{B} [voir Théorie des distributions, t. II, Hermann, Paris, 1951, pp. 55-59; MR 12, 833]; le dual de \mathcal{D}'_L est l'espace \mathcal{B} . L'intégrale d'une distribution T sommable est définie par $\int T_x dx = \langle 1, T \rangle$ (on a $1 \in \mathcal{B}$). Soit maintenant $T_{x,y}$ une distribution sur $X \times Y$, avec $X = R^m$, $Y = R^n$. Pour définir l'intégrale partielle $\int T_{x,y} dx$, on interprète $T_{x,y}$ comme distribution de x à valeurs dans \mathcal{D}'_Y . Or, d'après le „théorème des noyaux”, on a l'isomorphisme $\mathcal{D}'_{x,y} \cong L(\mathcal{D}_x, \mathcal{D}'_y)$; on a même $L_b(\mathcal{D}, E) \cong \mathcal{D}' \hat{\otimes} E$, pour tout espace localement convexe E complet (c'est là l'origine du concept d'espace nucléaire). On est alors conduit à nommer distributions à valeurs dans E les applications linéaires continues de \mathcal{D} dans E . On pose algébriquement, $\mathcal{D}'(E) = L(\mathcal{D}, E)$ et on y considère plusieurs variantes topologiques.

Plus généralement, soit \mathcal{K}' un espace de distributions ($\mathcal{E}' \subset \mathcal{K}' \subset \mathcal{D}'$, les immersions étant continues). On se propose d'associer à \mathcal{K}' un espace $\mathcal{K}'(E)$ de distributions à valeurs dans E . Il y a plusieurs façons d'interpréter la question. Si \mathcal{K}' est le dual d'un espace \mathcal{K} , on pose encore $\mathcal{K}'(E) = L(\mathcal{K}, E)$. D'autre part, on considère l'espace $\mathcal{K}'(E)$ des distributions $T \in \mathcal{D}'(E)$ scalairement dans \mathcal{K}' (c.a.d. telles que $\varphi' T \in \mathcal{K}'$ pour tout $\varphi \in E$), muni d'une topologie convenable. Sous certaines hypothèses, on trouve $\mathcal{K}'(E) \cong L_c(\mathcal{K}, E) \cong \mathcal{K}' \hat{\otimes} E$. Exemples: $\mathcal{S}'(E) \cong L_b(\mathcal{S}, E) \cong \mathcal{S}' \hat{\otimes} E$; une distribution $U_{x,y} \in \mathcal{D}'_{x,y}$ sera dite tempérée en x si $U_{x,y} \in \mathcal{S}'_x(\mathcal{D}'_y)$; on peut alors définir la transformation de Fourier de $U_{x,y}$ par rapport à x .

Soit $T \in \mathcal{D}'_L(E) = L(\mathcal{B}, E)$; on pose par définition $\int T_x dx = T''(1)$, où T'' est la bitransposée de T (appliquant \mathcal{B} dans E''). Si E est réflexif on a donc $\int T_x dx \in E$. En général, il en est ainsi, lorsque T est „strictement sommable” (l'auteur en donne la définition et des conditions suffisantes). Par exemple, étant $T_{x,y} \in \mathcal{D}'_{x,y}$, on dira que $T_{x,y}$ est partiellement sommable en x , si $T_{x,y} \in (\mathcal{D}'_L)_x(\mathcal{D}'_y)$ alors $\int T_{x,y} dx \in \mathcal{D}'_y$. Soit $T_{x,y} \in (\mathcal{D}'_L)_{x,y}$; alors: 1°) T est sommable en x ; 2°) $\int T_{x,y} dx$ est sommable en y ; 3°) on peut intervertir l'ordre des intégrations (théorème de Fubini).

Comme applications, on présente des définitions intégrales de la convolution, de la transformée de Fourier et de la transformée de Laplace par rapport à l'une des variables. Pour la transformation de Laplace, on utilise l'espace \mathcal{L}_+ des distributions $T = D^k[g(x)]$, où g est un entier quelconque et $g(x)$ une fonction numérique conti-

nue bornée sur R , nulle pour $x < 0$. On considère ensuite $\mathcal{L}_+(E)$, où E est un espace localement convexe complet quelconque. Si E est du type $(\mathcal{D}\mathcal{F})$, l'intégrale $\int e^{-px} T_x dx$ est une fonction de p , holomorphe dans un demi-plan $\operatorname{Re} p > k$, à valeurs dans E (transformée de Laplace de T); ce résultat se généralise au cas où E est limite projective d'espaces $(\mathcal{D}\mathcal{F})$. En particulier, on définit ainsi la transformée de Laplace de $T_{x,y} \in (\mathcal{L}_+)_x(\mathcal{D}'_y)$.

Il y a encore lieu de signaler la définition de certains accouplements $\int v(T, \varphi) dx$, avec $T \in \mathcal{D}'(E)$, $\varphi \in \mathcal{D}(F)$, v étant un opérateur bilinéaire sur $E \times F$; on applique ces notions à l'étude d'opérations algébriques telles que le produit multiplicatif, le produit direct et la convolution, généralisées au cas des distributions vectorielles. Le dernier exposé (no. 24) finit avec l'étude d'une variante du théorème de Künneth, relatif au produit tensoriel de deux „modules à dérivation” (dû à Grothendieck).

Note du reviewer: Les méthodes suivies par le reviewer [voir l'oeuvre reviewé ci-dessus] permettent d'envisager, sous un angle nouveau, certains aspects de la théorie des distributions considérés dans cet exposé. En particulier, elles permettraient de refaire l'étude des distributions vectorielles, considérées, localement, comme dérivées de fonctions vectorielles continues. *J. Sebastião e Silva.*

★ Grothendieck, A. Quelques points de la théorie des produits tensoriels topologiques. Segundo symposium sobre algunos problemas matemáticos que se están estudiando en Latino América, Julio, 1954, pp. 173-177. Centro de Cooperación Científica de la UNESCO para América Latina, Montevideo, Uruguay, 1954.

Il s'agit d'un bref exposé, où l'auteur présente quelques définitions et quelques résultats récents de la théorie des produits tensoriels topologiques, en se limitant à la considération d'espaces de Banach (cf. l'analyse de la Thèse de l'auteur, ci-avant). Une application u de E dans F est dite préintégrale gauche, si E est l'espace quotient d'un espace de Banach L , de façon que l'application $L \rightarrow F$ déterminée par u soit intégrale; u est dite préintégrale droite, si F est un sous-espace normé d'un espace C , de façon que l'application $E \rightarrow C$ définie par u soit intégrale; u est dite préintégrale, si elle est préintégrale gauche et préintégrale droite. On a le théorème fondamental suivant, considéré par l'auteur comme le plus profond de la théorie: l'application identique d'un espace de Hilbert sur lui-même est préintégrale. Ce théorème caractérise les espaces de Hilbert du point de vue vectoriel-topologique. On en déduit plusieurs conséquences qui forment les fondements d'une théorie achevée des applications linéaires entre espaces L^1 , L^2 , L^∞ . *J. Sebastião e Silva.*

Grothendieck, Alexandre. Sur les espaces (F) et (DF) . Summa Brasil. Math. 3 (1954), 57-123.

This paper is in a way a sequel to the paper of Dieudonné and Schwartz on spaces of type (F) and (DF) [Ann. Inst. Fourier, Grenoble 1 (1949), 61-101; MR 12, 417], for it deals with the solution of questions raised at the end of that paper. Some of the results of the present paper were referred to in a survey lecture by Dieudonné [Bull. Amer. Math. Soc. 59 (1953), 495-512; MR 15, 963]. The present paper, then unpublished, is listed as item 20 in the bibliography of Dieudonné's published lecture. Spaces of type (DF) include the strong duals of spaces of type (F) , and all spaces which are normed or which are inductive limits of normed spaces. The theorem which motivates the definition of spaces of type (DF) is the following (all spaces are supposed to be topological and

linear): Suppose E is locally convex and metrisable and that $\{A_n\}$ is a sequence of equicontinuous subsets of E'' such that the union of all the A_n is bounded. Then A is equicontinuous. Here the bidual E'' has the topology of uniform convergence on the equicontinuous subsets of E' . The theorem just mentioned implies that E'' is of type (F) provided that E is locally convex and metrisable.

A locally convex space H is said to be of type (DF) if it has a countable fundamental family of bounded subsets and if, for $A_n \subset H'$, the union A of the A_n ($n=1, 2, \dots$) is equicontinuous whenever each A_n is equicontinuous and A is strongly bounded. Thus the strong dual E' of E is of type (DF) if E is locally convex and metrisable.

A space E is called a t -space (tonnelé) if every weakly bounded subset of E' is equicontinuous; it is a quasi- t -space if "weakly" is replaced by "strongly" in the foregoing definition. E is called bornological if every convex circled set in E which absorbs all bounded subsets of E is a neighborhood of 0; this implies that E is a quasi- t -space. E is called "distinguished" if every bounded subset of E'' is contained in the weak adherence of a bounded subset of E (E regarded as a subset of E''). The following can be asserted: A (DF) space is a quasi- t -space if its bounded subsets are metrisable. If E is locally convex and metrisable, the following assertions are equivalent: (1) E is distinguished; (2) the strong dual $E'=H$ is a t -space; (3) H is bornological. If E is a locally convex space which is the strict inductive limit of a sequence of locally convex metrisable distinguished spaces, then E is distinguished and its strong dual is a bornological t -space.

Numerous examples of (DF) spaces (of functions) are mentioned, among them the spaces of locally holomorphic functions studied by Köthe and Grothendieck. It is conjectured that under fairly general conditions the space $L(E, H)$ of continuous linear mappings of E into H may be of type (DF) if E is of type (F) and H is of type (DF).

There is given an example, due to G. Köthe but hitherto unpublished, of a space E of type (F) whose strong dual is not bornological; hence E is not distinguished. It is shown as well that the bidual E'' contains a bounded subset which is not weakly relatively compact. Numerous counter-examples are given for questions relating to (LF) spaces in the previously cited paper of Dieudonné and Schwartz. For instance: A quotient space of a reflexive (LF) space can fail to be complete. The dual E' of a reflexive (LF) space can contain a non-closed subspace whose intersection with every weakly compact set is weakly compact.

In the last part of the paper the author introduces a definition of "quasi-normable" spaces. One way of expressing the definition is as follows: A locally convex space E is quasi-normable if to every neighborhood U of 0 corresponds another such neighborhood V such that if $\alpha > 0$ there exists a bounded set M for which $V \subset \alpha U + M$. The strong dual of an (F) space is quasi-normable. A metrisable and quasi-normable space is distinguished and its bidual has these same properties. Quasi-normability of a subspace is useful in a situation where one wants to extend simultaneously a whole sequence of linear functionals defined on the subspace, in such a way as to preserve in the extensions certain properties of the sequence. Quasi-normability is also related to weak compactness of linear mappings. In particular, a continuous linear mapping of a reflexive quasi-normable space into a Banach space is weakly compact. A Schwartz space (type (S)) is defined to be a quasi-normable space in which bounded sets are precompact. The property of being an

(S) space is inherited by subspaces, quotient spaces, and by products and inductive limits of spaces of type (S).

A. E. Taylor (Los Angeles, Calif.).

Sebastião e Silva, J. Sur une construction axiomatique de la théorie des distributions. Univ. Lisboa. Revista Fac. Ci. A. (2) 4 (1955), 79-186; 5 (1956), 169-170.

Définition axiomatique des Distributions, analogue à celle de H. König [Math. Nachr. 9 (1953), 129-148; MR 14, 1072]. Soit E un groupe abélien, Λ un semi-groupe d'homomorphismes de sous-groupes de E sur E , contenant l'identité. On suppose que chaque $\Phi \in \Lambda$ admet une inverse à droite Φ_d^{-1} avec $(\Phi\Psi)_d^{-1} = \Phi_d^{-1}\Psi_d^{-1} = \Psi_d^{-1}\Phi_d^{-1}$, et qu'à chaque Φ est associé un sous-groupe $\hat{N}(\Phi)$ de E , contenant le noyau $N(\Phi)$ (condition oubliée par l'auteur), et satisfaisant à certaines conditions. Alors il existe un sur-groupe \tilde{E} et un seul de E (à un isomorphisme près) et des prolongements uniques $\tilde{\Phi}$ des Φ en endomorphismes de \tilde{E} , de telle sorte que $(\Phi\Psi) \sim \tilde{\Phi}\tilde{\Psi}$, que $E \cap N(\tilde{\Phi}) = \hat{N}(\Phi)$, et que tout élément de \tilde{E} puisse s'écrire comme $\tilde{\Phi}(u)$ convenable, $\Phi \in \Lambda$, $u \in E$ (théorème I, p. 98). Si E est un espace vectoriel topologique, localement convexe séparée, si les Φ_d^{-1} sont continues et les $\hat{N}(\Phi)$ fermés, on introduira sur \tilde{E} la topologie localement convexe la plus fine pour laquelle les $\tilde{\Phi}$ soient continues de E dans \tilde{E} .

Soit Q un pavé fermé de R^n . Si E_Q est l'espace vectoriel normé des fonctions numériques continues sur Q , Λ le semi-groupe des dérivations partielles d'ordre quelconque en x_1, x_2, \dots, x_n , alors \tilde{E}_Q est l'espace des "distributions sur Q ". L'espace \mathcal{D}_Q des distributions sur l'ouvert Ω de R^n est alors défini comme la limite projective des \tilde{E}_Q pour les $Q \subset \Omega$. Sa topologie peut être définie par les suites convergentes, et il est en particulier bornologique (propriété démontrée indépendamment par A. Grothendieck [voir p. 85, théorème 10 de l'oeuvre analysée ci-dessus]). (Si Δ est l'adhérence d'un ouvert borné de R^n , l'auteur introduit l'espace des "distributions sur Δ " comme lorsque Δ est un pavé Q . Mais il y a une erreur page 131: si f est une fonction continue sur Δ , f est discontinue sur R^n , et $\mathfrak{F}_{x_i} f$ n'est pas nécessairement une fonction continue sur Δ , donc \mathfrak{F}_{x_i} n'est pas un inverse à droite. L'auteur me prie de signaler que cette erreur peut être redressée par un raisonnement plus compliqué; de toute façon cette notion n'est pas indispensable pour la suite, on peut se contenter des pavés Q). Enfin une méthode d'"analyse linéaire" des distributions permet d'écrire $T(x) = \int T(u) \delta(x-u) du$ en un sens convenable, et ceci permet de montrer que le dual de \mathcal{D}' est \mathcal{D} ; \mathcal{D}' étant par ailleurs réflexif, sa topologie définie ci-dessus est sa topologie usuelle de dual fort de \mathcal{D} .

L. Schwartz (Paris).

Iséki, Kiyoshi. Vector-space valued functions on semi-groups. III. Proc. Japan Acad. 31 (1955), 699-701.

[For parts I, II see same Proc. 31 (1955), 16-19, 152-155; MR 16, 1030; 17, 175.] $f(x)$ is an almost periodic (a.p.) function into a locally convex vector space E on a semi-group G , with unit. (a) For each neighborhood U of E and each triple $x, a, b \in G$, there is an element $x' \in G$ such that $f(cxd) - f(cax'bd) \in U$ for all $c, d \in G$. From this result is derived the existence and uniqueness of a Maak function $f(x, y)$ for $f(x)$, viz., a function such that (i) $f(x, y)$ is a.p. in x for every y , (ii) $f(x, 1) = f(x)$, (iii) $f(xa, ya) = f(x, y)$ for all $x, y, a \in G$. In outline, the proof runs as follows: From (a), for each U of E , $y \in G$, there is a $y' \in G$ such that $f(cd) - f(cy'y'd) \in U$. Set $f_0(x, y) = f(xy')$, and

then set $f(x, y) = \lim_{\alpha} f_{\alpha}(x, y)$. The existence of this limit and the uniqueness of the Maak function follow somewhat directly. Several misprints and ambiguities of presentation unfortunately obscure the contents of this note.

B. Gelbaum (Minneapolis, Minn.).

Tagamlicki, Ya. On a generalization of the concept of irreducibility. Ann. Univ. Sofia Fac. Sci. Phys. Math. Livre 1. 48 (1953/54), 69-85 (1954). (Bulgarian. Russian summary)

Let K be an additive abelian semigroup with a zero, admitting nonnegative real numbers as operators, and satisfying the usual compatibility conditions (i.e., K would be a linear space if subtraction were possible). Let P be a norm for K subject to the usual conditions. An element $a \neq 0$ in K is irreducible if $0 \neq a = b + c$ and $P(a) = P(b) + P(c)$ implies that $\lambda b = \mu c$ for non-negative numbers λ and μ such that $\lambda + \mu > 0$. Imposing some heavy further conditions on K , the author proves that irreducible elements exist. Applications are made to various known theorems on moments. [For earlier work in the same spirit, see Bulgar. Akad. Nauk. Izv. Mat. Inst. 1 (1953), 57-68, and the literature there cited; MR 15, 442.] E. Hewitt.

Haplanov, M. G. Spectrum of a matrix in an analytic space. Rostov. Gos. Univ. Uč. Zap. Fiz.-Mat. Fak. 32 (1955), no. 4, 3-8. (Russian)

After mentioning a number of properties of the space l_1 (all functions analytic in the unit circle) and l_{∞} (all functions whose power series have bounded coefficients) the author proves that any matrix M mapping l_{∞} into l_1 has a point spectrum. That is, the matrices $I - \lambda M$ have unique inverses except for, at most, a set of isolated λ with no finite limit point. The proof depends on the Fredholm theory of integral equations. D. C. Kleinecke.

See also: Valnberg, p. 718; Eremin, p. 723.

Banach Spaces, Banach Algebras

Ghika, Al. Polyédroides convexes et espaces vectoriels multiplement ordonnés. Com. Acad. R. P. Romîne 5 (1955), 311-315. (Romanian. Russian and French summaries)

A linearly bounded convex subset P of the real linear space E is called a "polyhedroid" provided there is a finite set X of extreme points of P such that $P = \bigcap_{x \in X} (P_x + x)$, where P_x is the convex cone $[0, \infty[(P - x)$. The following results are stated: (Theorems 1 and 2) For a subset C of E , the following two assertions are equivalent: (a) C is a polyhedroid for which $C = -C$ and $[0, \infty[C = E$. (b) There is on E a finite set $\{\leq_i\}$ of order relations, each compatible with the linear structure and each admitting an order-unit e_i , such that $-e_i \leq_j e_j \leq_i e_i$ for all i, j , and $C = \{x: \text{for all } i, -e_i \leq_i x \leq_i e_i\}$. (Theorem 3) Suppose (a) above is true and $C = \bigcap_{x \in X} (C_x + x)$, where X is a finite set of extreme points of C . Let F be the family of all sets of the form $tC + y$ for real t and $y \in E$. Then F has the binary intersection property if and only if for each $x \in X$ and each family T of translates of C_x , the intersection of T is either empty or a translate of C_x . For related results and definitions see Nachbin [Trans. Amer. Math. Soc. 68 (1950), 28-46; MR 11, 369]. [Proposition 1 is false as stated, but becomes correct if the plus signs are replaced by minus. Similar changes should be made later in the paper, though the symmetry assumptions render them not essential.]

V. L. Klee, Jr. (Los Angeles, Calif.).

Ghika, Al. Modules paranormés. Com. Acad. R. P. Romîne 5 (1955), 317-323. (Romanian. Russian and French summaries)

An algebra A over the real field is called "corpoidal" if A has a unit 1 and a finite system S of generators such that $S = -S$ and S is a multiplicative group having 1 as unit. If E is a left unitary module over A , an "S-norm" on E is a real-valued function ϕ which is positively homogeneous, subadditive, and invariant under S (i.e., $\phi(\alpha x) = \phi x$ for all $\alpha \in S, x \in E$). The S-norm ϕ is said to be "polyhedroidal" provided the set $C = \phi^{-1}[0, 1]$ is a polyhedroid with $C = -C$ and $[0, \infty[C = E$. [See the preceding review.] Several examples relevant to these notions are discussed, and a theorem is stated describing how polyhedroidal S-norms can be generated by certain polyhedroids in E .

V. L. Klee, Jr.

Ghika, Al. Prolongement des applications linéaires et continues dans des modules paranormés. Com. Acad. R. P. Romîne 5 (1955), 503-507. (Romanian. Russian and French summaries)

Ghika, Al. Nécessité de la condition de prolongement d'une application linéaire et continue dans les modules paranormés. Com. Acad. R. P. Romîne 5 (1955), 955-958. (Romanian. Russian and French summaries)

Let A be a corpoidal algebra (as in the preceding review), E and E' unitary modules over A with S-norms ϕ and ϕ' respectively, and T a continuous linear transformation into E' of a submodule of E (E and E' topologized by means of ϕ and ϕ' respectively). The principal result of the first paper is that if the family of all closed spheres in E' has the binary intersection property, then there is a continuous linear normpreserving extension of T to all of E . The second paper claims that for a given T , the binary intersection property in E' is necessary for the existence of an extension of the sort described, but presumably the author wishes to say that if the extension exists for each E and T as above, then the binary intersection property holds in E' . These results should be compared with those of Nachbin [Trans. Amer. Math. Soc. 68 (1950), 28-46; MR 11, 369]. In these papers, as in the two reviewed above, no proofs are given.

V. L. Klee, Jr. (Los Angeles, Calif.).

Thoma, Elmar. Darstellung von vollständigen Vektorverbänden durch vollständige Funktionenvektorverbände. Arch. Math. 7 (1956), 11-22.

Let R be a complete vector lattice. R is completely isomorphic to a complete vector lattice of real-valued functions if and only if the strongest possible distributive law holds for \vee and \wedge in R . Furthermore, an arbitrary complete vector lattice R can be uniquely decomposed as an order-preserving direct sum of subspaces B and B^* such that: 1) $a \in B$ (B^*) and $0 < b \leq |a| \Rightarrow b \in B$ (B^*); 2) B and B^* are complete vector sublattices of R ; 3) B is completely homomorphic to no complete function lattice except $\{0\}$; 4) B^* is completely isomorphic to a complete function lattice. [See in this connection also a paper of H. Bauer, S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. 1953, 89-117; (1954); MR 16, 50.] E. Hewitt (Seattle, Wash.).

Rado, Richard. Minimal points of convex sets in sequence spaces. Math. Z. 63 (1956), 486-495.

If E is a Banach space, E^* its adjoint, and C^* the unit cell of E^* , then C^* is weakly compact. It follows that each weakly closed subset X of E^* includes a point of minimum

norm, a result applied by W. W. Rogosinski [Math. Z. 63 (1955) 97-108; MR 17, 273] to linear equations in infinitely many variables. Now if E^* is not strictly convex, then X (even if convex) may include many points of minimum norm, and it is of interest to select those with additional minimizing properties. In the present paper, this is accomplished for E^* the space of bounded sequences, the resultant theorem leading to a new theorem on linear equations.

Let I be the set of all integers ≥ 0 , $(E_i)_{i \in I}$ a sequence of normed linear spaces, and S the linear space of all sequences $x = (x_i)_{i \in I}$ for which always $x_i \in E_i$ and $\|x\|_I < \infty$. (Here for $N \subset I$, $\|x\|_N = \sup\{\|x_i\|_{E_i}\}_{i \in N}$, where $\|x_i\|_{E_i}$ is the norm of x_i in E_i .) Let Rx be the set of all j for which $\{x_i\}_{i \in I}$ has a strictly increasing subsequence $> |x_j|$, and let $Lx = I \sim Rx$. There are an ordinal $\alpha \in [0, \omega_1[$ and a biunique order-reversing map $\rho_\alpha x$ on $[0, \alpha x[$ to $\{x_i\}_{i \in Lx}$. For $\nu \in [0, \alpha x[$, let $N_\nu x = \{i: |x_i| = \rho_\alpha x\}$, $L_\nu x = \bigcup_{\mu < \nu} N_\mu x$, $R_\nu x = I \sim L_\nu x$. With XCS , a "minimal sequence for X " is a point $z \in X$ such that for each $x \in X$, either $x = z$ on Lz and $\|x\|_{Rz} \geq \|z\|_{Rz}$ or there exists $\nu < \alpha z$ with $x = z$ on $L_\nu z$ and $\|x\|_{R_\nu z} > \|z\|_{R_\nu z}$. Now suppose that for each i , E_i is strictly convex and T_i a topology for E_i , and that X is a non-empty convex subset of S which is closed in the topology $\prod_{i \in I} T_i$. The author shows that if each E_i is finite-dimensional and T_i is its norm topology, then there exists a minimal sequence for X . [More generally, it is enough to assume that for each i , T_i is coarser than the norm topology on E_i , and the unit cell of E_i is compact under T_i . In § 9, the author seems to assert that finite-dimensionality may be replaced by completeness above, but a simple counterexample shows that some sort of compactness is needed.] V. L. Klee, Jr. (Los Angeles, Calif.).

Gurevič, L. A. On a basis in the space of continuous functions defined on a closed bounded set in n -dimensional space. Voronež. Gos. Univ. Trudy Fiz.-Mat. Sb. 27 (1954), 84-87. (Russian)

By using a projection onto the coordinate axes and a trivial modification of the Schauder construction for a basis of the continuous functions on a closed interval, the author produces a basis for the continuous functions over any compact set in finite-dimensional space. In view of the result of Baher [Dokl. Akad. Nauk SSSR (N.S.) 101 (1955), 589-592; MR 16, 1031] the result before us is an "applied" rather than a "pure" theorem, since Baher's paper shows that the continuous functions over any compactum have a basis. B. Gelbaum.

Krasnosel'skii, M. A.; and Rutickii, Ya. B. On a method of constructing N' -functions equivalent to the complementary ones to given N' -functions. Voronež. Gos. Univ. Trudy. Fiz.-Mat. Sb. 33 (1954), 3-17. (Russian)

In numerous problems connected with Orlicz space L_M^* there arises the need for the set of functions constituting L_N^* where M and N are complementary N' -functions. For most purposes, if an N_1 can be found such that the sets $L_{N_1}^*$ and L_M^* are identical, enough is accomplished. In this paper, the authors solve the problem of finding such an N_1 for a given M from a certain class of M .

Two N' -functions M_1 and M_2 are called equivalent ($M_1 \sim M_2$) if the sets $L_{M_1}^*$ and $L_{M_2}^*$ are the same. Criteria for equivalence of M_1 and M_2 are given and these range from such an elementary one as (a) $M_1(\alpha u) \leq M_2(u) \leq M_1(\beta u)$ for some $\alpha, \beta > 0$, all u sufficiently large, to so involved a test as (b): Let M_1, M_2, N_1, N_2 be N' -

functions, N_1 complementary to M_1 , $M_1(u) = \int_0^{|u|} p_1(t) dt$, $N_2(u) = \int_0^{|u|} q_2(t) dt$ (these representations always exist). Then if there is a set F on the real line R such that $m(R \setminus F) = 0$ and such that for some $b > 0$,

$$\lim \{p_1(q_2(v))/v \mid v \in F, v \rightarrow \infty\} = b,$$

then $M_1 \sim M_2$, $N_1 \sim N_2$.

By using the concept of "principal part" of an N' -function M , i.e., a function Q such that $\lim M(u)/Q(u) = a > 0$ as $u \rightarrow \infty$, and a class $\mathfrak{M} = \{R(u) \mid R(u) \neq 0 \text{ for large } u, R'(u) = r(u) \text{ exists for large } u, \text{ and } \lim u(r(u)/R(u)) = 0 \text{ as } u \rightarrow \infty\}$, the following theorem is derived: (Note: \mathfrak{M} contains all functions of the form

$$(\log u)^{\alpha_1} (\log \log u)^{\alpha_2} \cdots (\log \log \cdots \log u)^{\alpha_n},$$

$\alpha_1, \alpha_2, \dots, \alpha_n$ arbitrary.)

Let M and N be complementary N' -functions. Let $R(u) \in \mathfrak{M}$, and assume $M(u)R(u)$, resp. $N(u/R(u))R(u)$, are for large u equal to the N' functions Φ , resp. Ψ_1 . Let $M(u) = \int_0^{|u|} p(t) dt$, $N(u) = \int_0^{|u|} q(t) dt$, and assume $p_1(u) = p(u)R(u)$, and $q_2(u) = q(u/R(u))$ are monotone increasing for large u and that $\lim R(q(u/R(u)))/R(u) = b > 0$ as $u \rightarrow \infty$. Then $\Psi_1 \sim \Psi$, where Ψ is the complementary function of Φ .

More explicit forms for linear functionals over Orlicz spaces are derived. B. Gelbaum.

Rutickii, Ya. B. Application of Orlicz spaces for the investigation of certain functionals in L^2 . Dokl. Akad. Nauk SSSR (N.S.) 105 (1955), 1147-1150. (Russian)

If $f(x, u)$ is defined on $G \times R$, where G is a compact set in finite-dimensional space and R is the real line, and if f is continuous in u for each x and measurable in x for each u , then the operator $f(\phi) = f(x, \phi(x))$ is a mapping from a function space to its conjugate space, whereas $A(\phi) = \int_G K(x, y) f(y, \phi(y)) dy$ is an operator of a more general kind. Among others, the following results are to be found: (1) If the operator f maps a sphere of Orlicz space L_M^* into the class L_N , where M and N are complementary N' -functions, then f is the gradient of the functional $F(\phi) = \int_G dx \int_G f(x, \phi(x)) f(x, s) ds$. (2) If $\int_G \int_G M(Q(K(x, y))) dx dy < \infty$, where K is a positive definite kernel and M and Q are N' -functions, the former satisfying the Δ^2 -condition (i.e., $\limsup \{M(2u)/M(u) \mid u \rightarrow \infty\} < \infty$, and if $|f(x, u)| \leq b + aM(ku)$, then A defined in L_M^* is of the form HH^* , where H^*/H is the gradient of a weakly continuous functional in L^2 and H is a linear operator from L^2 into L_M^* . (3) If $\|K(x, y) - \sum_{i=1}^n \phi_i(x) \phi_i(y) / \lambda_i\|_p \rightarrow 0$, where $0 < \lambda_1 < \lambda_2 < \dots$ and ϕ_i are ON in L^2 , then the linear operator $B(\phi) = \int_G K(x, y) \phi(y) dy$ is of the form $B = HH^*$, where H is a linear completely continuous operator from L^2 into L^p . B. Gelbaum (Minneapolis, Minn.).

Krabbe, G. L. On the logarithm of a uniformly bounded operator. Trans. Amer. Math. Soc. 81 (1956), 155-166.

Making use of a construction of E. R. Lorch [same Trans. 49 (1941), 18-40; MR 2, 224] to break the spectrum of an operator U satisfying $\|U^n\| \leq K < \infty$, $n=0, \neq 1, \neq 2, \dots$, into two parts, the author constructs a logarithm for such an operator under the additional hypothesis that U acts in a reflexive space, and that neither $+$ nor $-$ belong to the point spectrum of U . The uniqueness and certain spectral-theoretic properties of such a logarithm are also investigated. J. Schwartz (New York, N.Y.).

Gohberg, I. C.; and Markus, A. S. On a characteristic property of the kernel of a linear operator. Dokl. Akad. Nauk SSSR (N.S.) 105 (1955), 893-896. (Russian)

Let A be a closed operator on a Banach space, and Γ a connected domain such that $A - \lambda I$ has a bounded inverse for $\lambda \in \Gamma$. The authors prove that, given y , the equation $Ax - \lambda x = y$ is solvable for all λ if and only if y is in the kernel of $A - \lambda_0 I$ for some $\lambda_0 \in \Gamma$. The kernel of an operator is defined as the intersection, over all positive n , of the ranges of A^n . The kernels of $A - \lambda I$ are the same for all $\lambda \in \Gamma$. If y is not in the kernel, the equation $Ax - \lambda x = y$ is solvable for, at most, a set of isolated points in Γ . The authors state an extension of this result to operator functions A_λ analytic on Γ .
D. C. Kleinecke.

Markus, A. S. On a characteristic property of the kernel of a linear operator. Dokl. Akad. Nauk SSSR (N.S.) 105 (1955), 1144-1146. (Russian)

The results of the paper reviewed above are extended to the case where the range of $A - \lambda I$ is closed and the null space of $A - \lambda I$ is finite dimensional.
D. C. Kleinecke.

Hurgen, Ya. L. On subrings of the ring of complex continuous functions. Moskov. Gos. Univ. Uč. Zap. 145, Mat. 3 (1949), 165-167. (Russian)

Let Z be the Banach algebra of all complex continuous functions on the disc ($|z| \leq 1$), and K a closed subalgebra of Z containing all functions analytic in $|z| < 1$. Then, $K = Z$ if K contains the real and imaginary parts of a function w which satisfies the following condition: if z' and z'' are distinct points of the disc, and $w(z') = w(z'')$, then there exists a curve Γ which separates the disc into two simply connected regions containing z' and z'' , respectively, and such that $w(z) \neq w(z')$ for $z \in \Gamma$. In particular, $K = Z$ if K contains the real or imaginary part of a non-constant analytic function.
M. Jerison.

Urbanik, K. On quotient-fields generated by pseudo-normed rings. Studia Math. 15 (1955), 31-33.

Soit R une algèbre sur le corps complexe C , commutative, unitaire, sans diviseur de 0, ayant une topologie localement convexe métrisable pour laquelle la multiplication soit continue. Sur le corps des fractions $Q(R)$, la convergence de Mikusiński est définie comme suit: p_n/q_n converge vers 0 pour $n \rightarrow \infty$ s'il existe $r \in R$ tel que $r p_n/q_n$ converge vers 0 dans R . Alors, s'il existe sur $Q(R)$ une topologie localement convexe métrisable, pour laquelle $x \rightarrow x^{-1}$ soit continue, et pour laquelle la convergence des suites soit celle de Mikusiński, R est le corps C (algébriquement et topologiquement). [En effet $Q(R)$ est un corps topologique localement convexe (multiplication continue, inverse continu), donc identique à C .]
L. Schwartz (Paris).

Matsumura, Yoshimi. Note on Shimoda's three sphere theorem. Kōdai Math. Sem. Rep. 7 (1955), 45-47.

The principal result is the following: Let E, E' be complex Banach spaces; let f on E to E' be defined and analytic when $R_1 \leq \|x\| \leq R_2$. For $R_1 \leq r_1 \leq R_2$ let

$$M_p(r) = \sup_{\|x\| \leq r} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \|f(e^{i\theta} x)\|^p d\theta \right\},$$

where p is a positive integer. Then $M_p(r)$ is an increasing function of r and $\log M_p(r)$ is a convex function of $\log r$. The proof depends upon the existence of an analytic function h_p of E on all to some complex Banach space,

with the property that $\|h_p(y)\| = \|y\|^p$ for all y . This forces h_p to be a homogeneous polynomial of degree p [by results of Shimoda, Proc. Japan Acad. 30 (1954), 718-720; MR 16, 1123]. There is no discussion of the existence of such a function h_p . Also, the theorem for nonintegral p remains uncertain.
A. E. Taylor.

Sobolev, V. I. On linear functional equations. Voronež. Gos. Univ. Trudy. Fiz.-Mat. Sb. 27 (1954), 43-46. (Russian)

Another proof of the Fredholm theorem for the equation $U(x) - x = y$, where U is a completely continuous operator on a Banach space with basis.

D. C. Kleinecke (Albuquerque, N.M.).

Feldkin, S. A. The operator of singular integration on a broken contour in spaces with a weight. Kišinev. Gos. Univ. Uč. Zap. 11 (1954), 19-27. (Russian)

The author considers the operator

$$(S\varphi)(t_0) = \{(\pi i)^{-1} \int_L \varphi(t)(t - t_0)^{-1} dt\},$$

where L is a finite set of arcs, on a class of Banach spaces admitting functions with poles at some of the ends of the arcs. The theorems concern the closure of the range of S and its deficiency.
D. C. Kleinecke.

Krasnosel'skiĭ, M. A. Some problems of nonlinear analysis. Uspehi Mat. Nauk (N.S.) 9 (1954), no. 3(61), 57-114. (Russian)

In this monograph the author summarizes and systematizes methods and results developed during the last fifty years in the field of nonlinear functional analysis. The earliest investigations of this kind were concerned with problems of classical analysis, nonlinear boundary-value problems, nonlinear integral and integro-differential equations, and their authors, Lyapunov, E. Schmidt, Lichtenstein and others, used the traditional methods of analysis. More recent investigations, in particular the author's contributions, are mostly directed toward nonlinear problems of general functional analysis and they are characterized by variational and topological methods. The literature in the field, although not old, is very extensive. Some 55 authors with more than 100 papers are listed in the bibliography and referred to in the monograph. Even so the author has restricted himself to a selection of topics which are closest to his own interests: existence and uniqueness of solutions for general nonlinear equations; methods of approximation; branching and spectral theory. Nonlinear differential equations and special nonlinear equations are not considered at all.

Chapter I takes the reader from nonlinear integral operators to general nonlinear operators. Specially mentioned are Uryson's operator $A\varphi(s) = \int_G K(s, t, \varphi(t)) dt$, Hammersteins' operator $A\varphi(s) = \int_G K(s, t) f(t, \varphi(t)) dt$ and Lyapunov's operator given by an integral power series whose typical term is

$$\int_G \dots \int_G K(s, t_1, \dots, t_p) \varphi^{p_1}(s) \varphi^{p_2}(t_1) \dots \varphi^{p_p}(t_p) dt_1 \dots dt_p.$$

These operators are considered as acting on C, L_p or some Orlicz space, whereas the general operators act on an abstract Banach space. They are studied as to their continuity, complete continuity, (Fréchet) differentiability, asymptotic closeness to linear operators (the linear operator B is asymptotically close to the nonlinear operator A if $\lim_{\|x\| \rightarrow \infty} \|A\varphi - B\varphi\| = 0$) and gradient property (whether gradient of a functional).

In Chapter II existence and uniqueness of solutions of equations like $\varphi = A\varphi$ and $\varphi = \lambda A\varphi + f$ are taken up as well as stability and approximation of the solutions. The guiding ideas in this area are successive approximations, fixed points of mappings, topological index of vector fields, homotopy of mappings, variational principles, and approximation by algebraic systems.

Chapter III which deals with the eigenfunctions of nonlinear operators contains most of the author's original contributions. The existence of eigenvectors, i.e. solutions $\varphi \neq \theta$ of $A\varphi = \lambda\varphi$ when $A\theta = \theta$ is proved by topological methods (using the index of the solution θ of $(I - \lambda A)\varphi = \theta$), by variational principles when A is the gradient of a weakly continuous functional, by use of the Lyusternik-Schnirelman category (if A is odd, i.e. $A(-\varphi) = -A\varphi$). The branching points are taken up next (μ_0 is a branching point of A if for given $\delta > 0$, $\varepsilon > 0$ there exist φ, μ such that $\varphi = \mu A\varphi$ and $\|\varphi\| < \delta$, $|\mu - \mu_0| < \varepsilon$); μ_0 can be a branching point of A only if μ_0 is a characteristic value of the linear operator B that is the derivative of A . The converse is, in general, not true, but several conditions are known under which the "principle of linearization" holds. The branch points of A are identical with the characteristic values of B . The spectrum of A , i.e. the set of eigenvalues of A , is also studied, in particular conditions are presented under which the spectrum contains intervals. Finally positive operators, i.e. operators that leave a cone invariant, and their eigenvectors are discussed, in particular the question when the latter form a "continuous branch of infinite length" in the invariant cone.

The ideas behind the proofs for the various assertions are presented, but no complete proofs are given in this monograph.
M. Golomb (Lafayette, Ind.).

See also: Jakubík, p. 728; Vainberg, p. 751; Matsushita, p. 761; Laugwitz, p. 762; Haplanov, p. 767.

Hilbert Space

Inoue, Sakuji. Simplification of the canonical spectral representation of a normal operator in Hilbert space and its applications. Proc. Japan Acad. 31 (1955), 694-698.

The author announces a sequence of results (with only one brief outline of a proof) about the unitary equivalence theory of a certain class of normal operators; a detailed treatment is promised for the near future. The results have to do with specializing spectral theory and multiplicity to the class in question. The presentation is highly condensed; as a result of this fact, together with serious linguistic difficulties, the reviewer is unable to understand the main theorems. It is not even quite clear exactly what class of normal operators is being treated. It is clear that their point spectrum is required to be non-empty; it is probably also assumed that their entire spectrum lies on a simple curve.
P. R. Halmos (Chicago, Ill.).

Itô, Takasi. On the commutativity of projection operators. Proc. Japan Acad. 31 (1955), 682-683.

The author shows that if P and Q are projections on a Hilbert space, then various algebraic relations between P and Q (i.e., $PQ = QP$, $P \leq Q$, $P = Q$) are equivalent to certain conditions described in terms of norms and in terms of the projection R whose range is the (closed) span of the ranges of P and Q . Typical result: $PQ = QP$ if and only if $\|Rx\|^2 \leq \|Px\|^2 + \|Qx\|^2$ for every vector x .

P. R. Halmos (Chicago, Ill.).

Sul'din, A. V. On linear operators in a direct product of Hilbert spaces. Kazan. Gos. Univ. Uč. Zap. 114, no. 2 (1954), 169-172. (Russian)

Let H be the Kronecker product of the Hilbert spaces H_1 and H_2 and let A and B be linear operators defined in H_1 and H_2 respectively. The author makes a few simple observations about the closure and permutability of $A \times I$ and $I \times B$ when A and B are not necessarily bounded and everywhere defined. Specifically, if A and B are closed, then the closures $\overline{A \times I}$ and $\overline{I \times B}$ of $A \times I$ and $I \times B$ exist. Moreover, if A is bounded and B is closed, then for all x in the domain of $\overline{I \times B}$, $\overline{I \times B}(A \times I(x))$ exists and equals $A \times I(\overline{I \times B}(x))$. Finally, if $A - \lambda I$ has a bounded inverse for some λ and B is closed, then for all x in the intersection of the domains of $A \times I$ and $I \times B$ such that $\overline{A \times I}(x)$ is in the domain of $\overline{I \times B}$ it is true that $\overline{I \times B}(x)$ is in the domain of $A \times I$ and $I \times B(A \times I(x)) = A \times I(B \times I(x))$. The note concludes with applications to the operators $i\partial/\partial x$, $i\partial/\partial y$, $\int_0^1 f(x, y)dy$.
G. W. Mackey.

Fichera, Gaetano. Su un metodo del Picone per il calcolo degli autovalori e delle autosoluzioni. Ann. Mat. Pura Appl. (4) 40 (1955), 239-259.

Let E be the inverse of a completely continuous (but not necessarily self-adjoint) operator on a Hilbert space S . E is defined on a dense subspace U of S . It is clear that the function

$$(1) \quad \mu(\lambda) = \min_{u \in U} \|Eu - \lambda u\|^2 / \|u\|^2$$

of the complex variable λ is non-negative and vanishes precisely at the eigenvalues of E . Hence, it has its minima at these eigenvalues. If u_1, u_2, \dots is a sequence of vectors in U complete in S , then the function

$$(2) \quad \mu_n(\lambda) = \min_{u = \sum_{i=1}^n c_i u_i} \|Eu - \lambda u\|^2 / \|u\|^2$$

approximates $\mu(\lambda)$. Picone's method consists of considering the minima of $\mu_n(\lambda)$ as approximations to the eigenvalues of E . This paper investigates convergence of this method.

The author shows that spurious minima of $\mu_n(\lambda)$ may occur. These are eliminated as follows: Let $u_n^{(n)}$ be a unit vector for which the ratio in (2) attains the value $\mu_n(\lambda)$. Let

$$(3) \quad e_n(\rho) = \limsup_{|\lambda| \leq \rho} \| (E^* - I)(E - I)u_n^{(n)} \|^2 - [\mu_n(\lambda)]^2,$$

where E^* is the adjoint of E and I the identity. Let $L_n^{(n)}$ be the set of minima of $\mu_n(\lambda)$ for which $|\lambda| \leq \rho$ and

$$(4) \quad [\mu_n(\lambda)]^2 \leq e_n(\rho).$$

The set $L_n^{(n)}$ is shown to converge to the set L_ρ of eigenvalues of E of magnitude at most ρ in the following sense. Given any $\varepsilon > 0$, there exists an $n(\varepsilon)$ such that for $n \geq n(\varepsilon)$ there is a point of $L_n^{(n)}$ within distance ε of each point of L_ρ and a point of L_ρ within ε of each point of $L_n^{(n)}$. Thus for a fixed n several points of $L^{(n)}$ may approximate the same eigenvalue or one point may approximate several eigenvalues. Let now $\lambda_n \in L^{(n)}$ and suppose that λ_n converges to an eigenvalue λ as $n \rightarrow \infty$. The authors show that there exists at least one subsequence of the sequence $u_{\lambda_n}^{(n)}$ having a strong limit u and that u is an eigenfunction corresponding to λ . Thus, Picone's method as modified by the author provides a means of approximating eigenvalues and eigenfunctions of a non-self-adjoint differential operator.
H. F. Weinberger.

Citlanadze, È. S. On the variational theory of eigenvalues of nonlinear operators in Hilbert space. Akad. Nauk Azerbaidžan. SSR. Trudy Inst. Fiz. Mat. 4-5 (1952), 90-97. (Russian. Azerbaijani summary)

This article gives proofs of results concerning eigenvalues of nonlinear operators which are Fréchet differentials of even completely continuous operators: the results were stated in an earlier paper of the author [C. R. (Dokl.) Acad. Sci. URSS (N.S.) 53 (1946), 307-309; MR 8, 386].
J. L. B. Cooper (Cardiff).

Citlanadze, È. S. Some questions of nonlinear operators, generated by a Fréchet differential in Hilbert space, and their application. Akad. Nauk Azerbaidžan. SSR. Trudy Inst. Fiz. Mat. 4-5 (1952), 98-105. (Russian. Azerbaijani summary)

It is shown that a completely continuous operator generated as the Fréchet differential of a weakly continuous operator has a characteristic element on the unit sphere in Hilbert space [cf. the article cited in the preceding review].
J. L. B. Cooper (Cardiff).

Jamison, S. L. On analytic normal operators. Proc. Amer. Math. Soc. 5 (1954), 288-290.

By comparison of coefficients in the equation $A(z)A^*(z) = A^*(z)A(z)$, the author proves that if $A(z)$ is an analytic operator-valued function whose value is normal for $|z| < \rho$, then $A(z_1)A(z_2) = A(z_2)A(z_1)$. Consequently, there exists a Hermitian operator H and a function $f(z, \lambda)$

analytic in $|z| < \rho$ for each real λ , such that $A(z) = f(z, H)$.
J. Schwartz (New York, N.Y.).

Nussbaum, A. E. The Hausdorff-Bernstein-Widder theorem for semi-groups in locally compact Abelian groups. Duke Math. J. 22 (1955), 573-582.

Let an abelian semi-group S containing the unit 0 be imbedded in a locally compact abelian group G in such a way that i) S is measurable with respect to the Haar measure of G , and ii) every non-empty open set in S has non-zero measure. A real-valued continuous function $f(x)$ on S is called completely monotonic in S if and only if

$$\prod_i (I - T_{a_i})^{n_i} f(x) \geq 0 \quad (T_a f(x) = f(x+a))$$

for all integers $n_i \geq 0$ and all $a_i \in S$. The set M of functions $g(x) = \sum_i \alpha_i f(x+a_i)$ constitutes a pre-Hilbert space by the inner product $(g, h) = \sum_i \sum_j \alpha_i \beta_j f(x_i + y_j)$ (here $h(x) = \sum_j \beta_j f(x+y_j)$). Thus the translation operators $\{T_a\}$ constitute a uniformly bounded semi-group of self-adjoint operators acting on the Hilbert space H , obtained from M by the usual process of completion. By virtue of the simultaneous spectral resolution of $\{T_a\}$, based upon the theory of normed rings, it is proved that $f(x)$ on S is completely monotonic if and only if it can be represented as $f(x) = \int_S \chi(x) d\alpha(\chi)$, where $\chi(x)$ is a character on S and $d\alpha(\chi)$ is a non-negative finite measure on the convex subset $S^0 = \{\chi | 0 \leq \chi(x) \leq 1\}$ of the character space.
K. Yosida (Tokyo).

See also: Loo and Kwan, p. 791.

TOPOLOGY

Aquaro, Giovanni. Funzioni reali uniformemente separate negli spazii uniformi ed applicazioni agli spazii normali. Ann. Mat. Pura Appl. (4) 39 (1955), 401-409.

Let E be a space with a uniform structure (no separation axiom assumed). Two real-valued functions f and g on E are said to be uniformly separated in E if for every $\epsilon > 0$ there is an entourage V of the diagonal in $E \times E$ such that $f(x') < g(x'') + \epsilon$ for all $(x', x'') \in V$. Theorem: If f and g are uniformly separated, then there is a uniformly continuous function h on E such that $f \leq h \leq g$. This fact is a generalization of, and can be used to prove, a theorem on separation of semicontinuous functions in normal spaces, due to Katětov [Fund. Math. 38 (1951), 85-91; 40 (1953), 203-205; MR 14, 304; 15, 640] and Tong [Duke Math. J. 19 (1952), 289-292; MR 14, 304].
E. Hewitt.

Mostert, Paul S. Sections in principal fibre spaces. Duke Math. J. 23 (1956), 57-71.

A Hausdorff space X on which a topological group G acts effectively without fixed points is said to be a principal fibre space if the orbit space B is again Hausdorff and the map $(x, xg) \rightarrow g$ from the subset $\{(x, xg) | x \in X, g \in G\}$ to G is continuous. One of the main results is that if B is locally compact then (X, B, ϕ, G) , where ϕ is the projection $X \rightarrow B$, is a principal fibre bundle. Applications: $X =$ locally compact topological group, G closed subgroup, $B = X/G$ of finite dimension, then (X, B, ϕ, G) is a fibre bundle.

A locally compact, locally connected Hausdorff space X of finite dimension which admits a locally compact transitive transformation group G satisfying the second countability axiom is a coset space of a Lie group. Finally, a theorem of Borel and Serre to the effect that a fibre space with a locally compact paracompact contractible base is trivial.
W. T. van Est (Utrecht).

Hayashi, Yoshiaki. Corrections to "On the dimension of topological spaces." Math. Japon. 3 (1955), 136.
See Math. Japon. 3 (1954), 71-84; MR 16, 1139.

Lelek, A. Une propriété de dualité équivalente à celle de Janiszewski. Bull. Acad. Polon. Sci. Cl. III. 3 (1955), 585-588.

Let X denote a (compact) metric continuum. X has property "J" if and only if for each two continua C_1 and C_2 in X such that $C_1 \cap C_2$ is not connected, $X - (C_1 \cup C_2)$ is not connected. X has property "Z" if and only if for each continuum C in X and each connected open subset R of X such that both $X - C$ and $X - R$ are connected, $C - R$ is connected if and only if $R - C$ is connected. The author proves that if X is locally connected, then it has property J if and only if it has property Z. Janiszewski proved that among those compact metric locally connected continua without any cut points the two-sphere is characterized by property J.
E. Dyer (Baltimore, Md.).

Harrold, O. G., Jr. A theorem on disks. Proc. Amer. Math. Soc. 7 (1956), 153-154.

An arc in E^3 (three space) is said to be semi-rectifiable if in some topological representation of the unit interval on it one of the coordinate functions is of bounded variation. The author proves that if J is a simple closed curve in E^3 and there is a homeomorphism of E^3 onto itself carrying J onto a simple closed curve containing a semi-rectifiable arc, then there is a tame disc D in E^3 such that (i) $D \cap J$ is a point in the interior of D and (ii) the boundary of D links J . R. H. Bing has given an example of a simple closed curve in E^3 which fails to have properties i) and ii) for any disc.
E. Dyer (Baltimore, Md.).

Shirai, Tameharu. Prolongation of the homeomorphic mapping. Proc. Japan Acad. 31 (1955), 147-151.

The author proves the following theorem: Any homeomorphism between two compact subsets of 1-dimensional Euclidean spaces contained in a 2-dimensional Euclidean space can be extended to a homeomorphism of the 2-dimensional Euclidean space onto itself. *W. T. Puckett.*

Williams, R. F. Local properties of open mappings. Duke Math. J. 22 (1955), 339-345.

Let f be an open mapping of a compact metric space X onto Y . All four of the author's theorems are of the form: If Y has a certain property and the inverses of points of Y have another, then X has a related property. Two of the theorems are: (1) If Y is locally connected at each point of a dense subset and $f^{-1}(y)$ is locally connected on a dense subset of itself for each $y \in Y$, then X is locally connected at each point of some dense G_δ subset. (2) If X is in the plane, Y is locally connected, and $f^{-1}(y)$ is locally connected for each $y \in Y$, then X is locally connected at each point of some dense open subset. The author shows by example that the hypothesis X is in the plane cannot be replaced by X is a continuum in ordinary 3-space. *W. T. Puckett (Los Angeles, Calif.)*

Taïmanov, A. D. On open mappings of Borel sets. Mat. Sb. N.S. 37(79), 293-300 (1955). (Russian)

Ce travail concerne quelques théorèmes sur les applications ouvertes qui sont multiformes dénombrablement ou multiformes finiment ou isolées. Théorème 1: Soit \mathfrak{M} une famille d'ensembles qui est topologique et σ -additive. Alors, une image continue d'un ensemble de \mathfrak{M} est celle ouverte d'un autre ensemble de \mathfrak{M} . Théorème 2: Soit \mathfrak{M} une famille de sous-ensembles de l'espace hilbertien \mathfrak{H} , qui est borelienne et contient tous les ensembles boreliens de \mathfrak{H} . Alors, si \mathfrak{M} est topologique, elle est aussi invariante par rapport aux applications ouvertes et isolées. Théorème 3: Soit f une application ouverte et multiforme finiment sur X . Alors, la classe de $f(B)$, où B est un ensemble borelien de X , ne dépasse pas celle de B . D'après le théorème 1, tout ensemble borelien est une image ouverte et multiforme dénombrablement d'un ensemble $F_\sigma \cap G_\delta$. D'après le théorème 2, la classe d'un ensemble borelien n'est pas augmentée par toute application ouverte et isolée. Ce sont des réponses pour une conjecture de F. Hausdorff [Fund. Math. 23 (1934), 279-291] et conduisent un résultat de Mlle. L. Keldych [C.R. (Dokl.) Acad. Sci. URSS (N.S.) 49 (1945), 622-624; MR 8, 16]. Encore, d'après le théorème 2, la dimension d'un espace n'est pas augmentée par toute application ouverte et isolée [voir P. S. Alexandrov, C. R. (Dokl.) Acad. Sci. URSS (N.S.) 4 (1936), 295-299]. *M. Kondó (Paris).*

Scorza Dragoni, Giuseppe. Una dimostrazione del teorema di Brouwer sulle traslazioni piane generalizzate. Ann. Mat. Pura Appl. (4) 39 (1955), 1-10.

Let t be an orientation-preserving fixed-point free automorphism of the plane E_2 . Let λ_0 be a translation arc, meaning a simple arc such that $\lambda_0 \cap t\lambda_0$ is a common end-point. It is known that the trajectory σ_0 , union of the images of λ_0 under positive and negative powers of t , is a topological image of the real line. The accumulation points of σ_0 which are not in σ_0 separate E_2 into connected components one of which contains σ_0 and is separated by σ_0 into two connected open sets Σ_0 and Σ_0' . The theorem which is proved is that there exists a closed topological image l of a half line, originating at a point R on λ_0 and such that $l - RC\Sigma_0$, $l \cap t(l) = \emptyset$. There also exists such an l

for Σ_0' . Assuming that l exists, let RR' be an arc of l . Then λ_0 could obviously be modified so as to skirt around RR' . The new translation arc λ_1 has an l which now begins with R' on λ_1 instead of R , and λ_1 can itself be modified. This construction of the sequence $\lambda_0, \lambda_1, \dots$ is based on l . The author's proof of the theorem follows the reverse process. Namely the construction of l is based on a sequence $\lambda_0, \lambda_1, \lambda_2, \dots$ which has to be constructed independently. It can be assumed that λ_0 is polygonal; each λ_i is obtained by the modulo 2 addition of a suitable triangle to λ_{i-1} . *P. A. Smith (New York, N.Y.).*

Wenjen, Chien. Quasi-equicontinuous sets of functions. Proc. Amer. Math. Soc. 7 (1956), 98-101.

A sequence $\{f_n\}$ of functions from a topological space X to a metric space Y (with metric ρ) is said to be ε -related at a point x of X if, for any $\varepsilon > 0$, there is a neighborhood $U(x)$ of x such that corresponding to each $x' \in U(x)$, there is a positive number N (depending on ε, x , and x') such that $\rho[f_n(x), f_n(x')] < \varepsilon$ for all $n > N$. A family F of continuous functions on X to Y is called quasi-equicontinuous if for every infinite subset Q of F and at every point x of X , there is a sequence $\{f_n\}$ contained in Q which is ε -related at x .

The author obtains the following generalization of the Ascoli theorem. Let X be locally separable, Y be metric, and let Y^X denote the set of all continuous functions from X to Y . A necessary and sufficient condition that a subset F of Y^X be compact in the point-open topology on Y^X is that: (1) F is closed in Y^X . (2) For each $x \in X$, $F(x) = \bigcup_{f \in F} f(x)$ is compact. (3) F is quasi-equicontinuous. *M. Henriksen (Lafayette, Ind.).*

van Rootselaar, B. On the mapping of spreads. Nederl. Akad. Wetensch. Proc. Ser. A. 58=Indag. Math. 17 (1955), 557-563.

van Rootselaar, B. Generating schemes for full mappings. Nederl. Akad. Wetensch. Proc. Ser. A. 58=Indag. Math. 17 (1955), 646-649.

Representation of the species of full mappings of finitary spreads into (onto) spreads or finitary or ω -spreads by ω -spreads; also under some uniformity condition. The same for unions of spread directions. Conditions for mappings of spread directions which are necessary and sufficient in order to generate full mappings of the spreads. *H. Freudenthal (Utrecht).*

Ringel, Gerhard. Über drei kombinatorische Probleme am n -dimensionalen Würfel und Würfelgitter. Abh. Math. Sem. Univ. Hamburg 20 (1955), 10-19.

If n is a power of 2, then the n -dimensional lattice can be decomposed into n Hamiltonian paths and the edges of the n -dimensional cube can be decomposed into $\frac{1}{2}n$ Hamiltonian circuits. For any n th graph determined by the vertices of the n -dimensional cube can be drawn without intersection of edges on an orientable surface of genus $(n-4)2^{n-3}+1$ and on no surface with smaller genus. If n is odd, then the $n(n-1)2^{n-3}$ sides of the n -dimensional cube can be divided into $\frac{1}{2}(n-1)$ classes with $n2^{n-3}$ members in such a way that the members of each class taken together form a closed orientable surface containing every edge of the cube. *G. A. Dirac.*

Dynkin, E. B.; und Uspenski, W. A. Mathematische Unterhaltungen. I. Mehrfarbenprobleme. Deutscher Verlag der Wissenschaften, Berlin, 1955. viii+65 pp. DM 5.10.

Translation of part I (pp. 13-47, 181-209) of Dynkin

and Uspenskii's *Matematicheskie besedy* [Gostehizdat, Moscow, 1952; MR 14, 455].

See also: Sibirskii, p. 718; Ishii, p. 720; Kamel, p. 704; Tutte, p. 708.

Algebraic Topology

Heller, Alex. Homotopy resolutions of semi-simplicial complexes. *Trans. Amer. Math. Soc.* 80 (1955), 299-344.

But du travail: classification homotopique des espaces avec groupes d'opérateurs, notamment des espaces fibrés principaux; recherche d'espaces classifiants pour un groupe topologique donné. Extension de la théorie de Postnikov et Zilber au cas des espaces avec groupes d'opérateurs.

Méthode: on travaille sur les complexes singuliers des espaces et des groupes; en fait, la théorie est développée pour les complexes semi-simpliciaux complets (CSS-complexes) au sens d'Eilenberg-Zilber.

Chap. I: rappel sur les CSS-complexes (très jolie exposition). Un CSS-groupe est un CSS-complexe Γ avec une CSS-application $\Gamma \times \Gamma \rightarrow \Gamma$ qui fait de chaque Γ_q un groupe; on définit la „composante connexe de l'unité”, qui est un CSS-groupe 0Γ ; Γ est „discret” si 0Γ est réduit à l'élément neutre en chaque dimension. Si Γ est un CSS-groupe, on a la notion de Γ -complexe: c'est un CSS-complexe X avec une CSS-application $\Gamma \times X \rightarrow X$ qui fait que Γ_q opère (à gauche) dans X_q . On définit alors le quotient X/Γ ; pour toute CSS-application $Y \rightarrow X/\Gamma$, on a un Γ -complexe „induit” Z , tel que $Z/\Gamma = Y$. Si aucun simplexe de X n'est fixe par aucun élément de Γ autre que l'identité, on dit que X est Γ -fibré (Γ -bundle): ceci correspond à la notion d'espace fibré principal.

Si Γ opère dans un groupe discret Π , on a groupe de „chaînes équivariantes” $\text{Hom}_\Gamma(C(X), \Pi)$, d'où des groupes de cohomologie équivariante $H_\Gamma^n(X; \Pi)$. [Note du rapporteur: pour l'homologie équivariante, la définition des chaînes équivariantes ne semble pas la bonne: p. 310, ligne 10*, il faudrait $(\gamma\sigma) \otimes p - \sigma \otimes (\gamma^{-1}p)$ au lieu de $(\gamma\sigma) \otimes p - \sigma \otimes (\gamma p)$].

Chap. II: groupes d'homotopie et obstructions. Les groupes d'homotopie d'un CSS-complexe X sont définis axiomatiquement (voir ci-dessous); ils n'existent pas nécessairement, mais s'ils existent leurs propriétés axiomatiques les déterminent à un isomorphisme près. Rappelons que si X est un CSS-complexe, le q -squelette X^q est le CSS-complexe engendré par X_q . Voici alors la définition des groupes d'homotopie: X étant supposé „homotopiquement connexe” (i.e.: deux sommets quelconques appartiennent à une même arête), un „système d'homotopie” est une suite de groupes abéliens $(\pi_1, \dots, \pi_q, \dots)$ et une opération naturelle c qui, à tout CSS-complexe K et à toute CSS-application $f: K^q \rightarrow X$ ($q \geq 1$), associe un cocycle normalisé $c_f \in Z^{q+1}(K; \pi_q)$ de manière à satisfaire à: (i) $c_f = 0$ si et seulement si f se prolonge à K^{q+1} ; (ii) pour tout sous-complexe LCK et toute q -chaîne normalisée h , il existe $f': K^q \rightarrow X$ qui coïncide avec f sur $K^{q-1} \cup L^q$ et satisfait à $c_{f'} - c_f = \delta h$.

Le système d'homotopie, s'il existe, est un foncteur covariant de X , et un invariant du type d'homotopie. La théorie classique des obstructions se développe alors. L'auteur donne aussi une théorie des obstructions pour les applications Γ -équivariantes. Il généralise la notion de complexe minimal d'Eilenberg-Zilber; complexe Γ -minimal.

Chap. III: pour chaque CSS-groupe Γ , l'auteur définit explicitement un CSS-groupe $U(\Gamma)$ et une injection $\Gamma \rightarrow U(\Gamma)$; $U(\Gamma)$ est foncteur covariant de Γ , et possède un système d'homotopie dont tous les π_q sont nuls. Le quotient $U(\Gamma)/\Gamma$ est un „espace classifiant”. L'auteur étudie ensuite, de son point de vue, les complexes d'Eilenberg-MacLane $K(\Pi, n)$. Plus généralement, soit Γ un CSS-groupe opérant dans le groupe abélien discret Π ; alors Γ opère dans $K(\Pi, n)$, et on a un produit croisé $K(\Pi, n) \cdot \Gamma$ („splitting extension”), noté $\Gamma(\Pi, n)$; $K(\Pi, n)$ en est un sousgroupe normal, le quotient étant Γ . L'auteur étudie les complexes $\Gamma(\Pi, n)$ -fibrés: un Γ -complexe B étant donné, les classes de complexes $\Gamma(\Pi, n)$ -fibrés X tels que $X/K(\Pi, n) = B$ sont en correspondance biunivoque avec les éléments du groupe de cohomologie équivariante $H_\Gamma^{n+1}(B; \Pi)$; tout ceci, sous réserve que $0X$ et Γ aient des systèmes d'homotopie.

Chap. IV: un Γ -fibré homotopiquement séparé („homotopically segregated”) est une suite $\mathbb{K} = (0K, \dots, qK, \dots)$ de Γ -fibrés avec homotopie, tels que: (S1) $0K$ soit asphérique; (S2) pour $q \geq 1$, $K(\pi_q(qK), q)$ opère dans qK , qK étant $K(\pi_q(qK), q)$ -fibré; (S3) $qK/K(\pi_q(qK), q) = q^{-1}K$; alors $\pi_r(q^{-1}K) = 0$ pour $r \geq q$.

On définit $(\lim \mathbb{K})^q = rK^q$ pour $r \geq q$, d'où un complexe $(\lim \mathbb{K})$, qui est Γ -fibré; si $0K$ est Γ -minimal, il en est de même de $\lim \mathbb{K}$; alors, si X est un Γ -fibré avec homotopie et si $\xi: (\lim \mathbb{K}) \rightarrow X$ est Γ -équivariante, les 3 conditions suivantes sont équivalentes: ξ est une homotopie-équivalence; ξ définit des isomorphismes $\pi_q(\lim \mathbb{K}) \sim \pi_q(X)$ pour $q \geq 1$; ξ est un isomorphisme sur un sous-complexe minimal de X . Il s'ensuit que, X étant donné, le couple (\mathbb{K}, ξ) , s'il existe, est unique à un isomorphisme près. Le théorème fondamental 15.4 affirme l'existence d'un tel couple (\mathbb{K}, ξ) , pourvu que 0Γ ait un système d'homotopie; (\mathbb{K}, ξ) s'appelle une „résolution homotopique” de X . La fin du mémoire est consacrée au cas particulier où le groupe Γ est „discret”; alors il est inutile de supposer que X soit Γ -fibré: il suffit que X soit un Γ -complexe. Soit $R_q(X)$ le quotient de X obtenu en identifiant deux simplexes qui ont le même q -squelette: si X possède un système d'homotopie, $R_q(X)$ aussi; et si en outre X est Γ -minimal, les $R_q(X)$ fournissent une résolution homotopique de X .

H. Cartan (Paris).

Toda, Hiroshi. Complex of the standard paths and n -ad homotopy groups. *J. Inst. Polytech. Osaka City Univ. Ser. A.* 6 (1955), 101-120.

Toda expounds his theory of standard paths. Except for some results on homotopy groups of spheres, which are to come in a sequel, this supersedes Proc. Japan Acad. 29 (1953), 299-304 [MR 15, 732]. “Reduced product spaces” [James, Ann. of Math. (2) 62 (1955), 170-197; MR 17, 396] are included as a special case. The theory is used to prove the following theorem about $(n+1)$ -ads.

An $(n+1)$ -ad is a space, X , with subspaces X_1, \dots, X_n whose intersection contains a basepoint X_0 . Let Δ be an n -simplex, and let F be the function space of maps of Δ into X such that the i th face of Δ goes into X_i ($0 \leq i \leq n$). Set $\pi_n(F) = \pi_{n+n}(X; X_1, \dots, X_n)$. Let X be a CW-complex, and let $X_i = X - (Y_1 - Y_i)$ ($1 \leq i \leq n$), where Y, Y_i are subcomplexes of X , whose union is X , such that $Y_i \cap Y_j = Y$ ($i \neq j$). Suppose that (Y_i, Y) is 2-connected and that Y is simply-connected. Theorem: Suppose that $H_p(Y_i, Y) \in C$ for $p \leq q_i$, where C is a class of abelian groups in the sense of Serre; then $\pi_{q+1}(X; X_1, \dots, X_n)$ is C -isomorphic to the direct sum of $(n-1)!$ copies of $H_{q+1}(Y_1, Y) \otimes \dots \otimes H_{q+1}(Y_n, Y)$, where $Q = \sum q_i$.

Standard path theory goes as follows. Let L be an FM-complex (a CW-complex with a certain kind of product). Then Toda defines a complex $K=B(L)$ by an identification $d: L \times I \rightarrow K$, where I is the interval $0 \leq t \leq 1$, such that $d(L \times I) = e_0 \in K$. He also defines a contractible complex M by an identification $\bar{d}: L \times I \rightarrow M$ such that $\bar{d} = \phi \bar{d}$, where ϕ maps M onto K . If $\bar{d}(y, t) = x$, say, where $y \in L$ and $t \in I$, then a standard path $l_x: I \rightarrow K$ is defined by $l_x(u) = \bar{d}(y, t + u - tu)$, $t \in I$. Let K' be a subcomplex of K , and let $\Omega(K, K')$ be the space of paths in K which start in K' and end at e_0 . Then $\omega(K, K') = \phi^{-1}(K')$ is a subcomplex of M , and $x \rightarrow l_x$ determines a non-singular map $i: \omega(K, K') \rightarrow \Omega(K, K')$. Theorem: i induces an isomorphism of the homotopy groups. Thus, the standard paths l_x are sufficient to carry all the usual homotopy invariants of the total space of paths. It turns out that \bar{d} maps $L \times 0$ homeomorphically onto $\omega(K, e_0)$, so that L itself is to be regarded as the complex of standard loops in K .

Let K be a CW-complex. We say that K admits standard paths if there is an FM-complex L such that $K=B(L)$. Let X be a simply-connected space and let X' be a simply-connected subspace. Theorem: There is a CW-complex K , which admits standard paths, and a subcomplex K' with a map $f: (K, K') \rightarrow (X, X')$ which induces an isomorphism of the homotopy sequences.

I. M. James (Princeton, N.J.).

Yokota, Ichiro. On the cell structures of $SU(n)$ and $Sp(n)$. Proc. Japan Acad. 31 (1955), 673-677.

The author shows that $SU(n)$ and $Sp(n)$ are cell complexes composed of 2^{n-1} and 2^n cells respectively. In the

construction, a fundamental role is played by so-called primitive cells, which are homeomorphic to the interior of the first suspended space of the $(2k-2)$ -dimensional complex projective spaces, with $n \geq k \geq 2$, in the case of $SU(n)$; and to the interior of the third suspended space of the $(4k-4)$ -dimensional quaternion projective spaces, with $n \geq k \geq 1$, in the case of $Sp(n)$. He deduces that the torsion groups are zero for all dimensions, in both cases, and that the Poincaré polynomials are $(1+t^2)(1+t^4) \cdots (1+t^{2n-2})$ and $(1+t^2)(1+t^4) \cdots (1+t^{4n-4})$, respectively. The author states that full details will be published later.

G. Papy (Brussels).

Ringel, Gerhard. Wie man die geschlossenen nichtorientierbaren Flächen in möglichst wenig Dreiecke zerlegen kann. Math. Ann. 130 (1955), 317-326.

By a triangulation of a closed surface is meant a division of the surface into triangles in such a way that two triangles either have no point or one vertex or one edge in common. δ_g denotes the minimum number of triangles required to triangulate a closed one-sided surface of genus g , i.e. there exists a triangulation of such a surface in which these are δ_g triangles but there exists no triangulation of such a surface in which these are fewer than δ_g triangles. It is shown that $\delta_2=16$, $\delta_3=20$, and if $g \neq 2, 3$, then $\delta_g=2[\sqrt{6g+2.5}]+2g$. The method used follows on an earlier paper by the author [Math. Ann. 127 (1954), 181-214; MR 16, 58], and it turns out again that the patterns or surfaces constructed are one-sided.

G. A. Dirac (Vienna).

See also: Novikov, p. 706; Martinelli, p. 785; Vesentini, p. 786.

GEOMETRY

★Brisac, Robert. Exposé élémentaire des principes de la géométrie euclidienne. Gauthier-Villars, Paris, 1955. 77 pp. 1200 francs.

This little book is intended for high-school teachers and superior high-school students. Consequently, reasons of simplicity require at times redundant postulates. Nevertheless, the novel approach will interest even specialists.

Points are undefined, lines and planes are point sets whose properties are fixed by axioms. Among these the emphasis on line elements (a point and an open ray issuing from it) and plane elements (an open half-plane with a line element on its boundary) is worth mentioning. Then the group concept is introduced. Motions are one-to-one mappings of the space on itself. The motions are, of course, required to form a group. The principal further requirements are: preservation of betweenness, the existence of exactly one motion taking one of two given plane elements into the other, the existence of a motion interchanging two given points, the existence of a motion interchanging the legs of a given angle. On the basis of these axioms involutonic motion and perpendicularity are studied. Then the Parallel Axiom is added.

After a discussion of Archimedean groups and measures in such groups, length is treated on the basis of an axiom requiring the translations along a line to form an Archimedean group. Finally a continuity axiom is introduced. A discussion of coordinate and scalar products follows. There are three appendices: the first on orientation, the second on the analytic representation of motions, and the third gives some other applications of measures in Archimedean groups, for instance to the definition of logarithms.

H. Busemann (Los Angeles, Calif.).

Goormaghtigh, R. Couples de polygones bordés de parallélogrammes. Mathesis 64 (1955), 340-344.

★Argunov, B. I.; i Balk, M. B. Geometričeskie postroeniya na ploskosti. Posobie dlya studentov pedagogičeskikh institutov. [Geometrical constructions in the plane. A method for students of teacher's colleges.] Gosudarstv. Učebno-Pedagog. Izdat., Moscow, 1955. 269 pp. 5.50 rubles.

★Barchanek, C. Lehr- und Übungsbuch der darstellenden Geometrie. Bearbeitet von Emil Ludwig und Josef Laub. 9te Aufl. Hölder-Pichler-Tempsky, Wien, 1954. v+222 pp. DM 8.00.

Die 9. Auflage des in Österreich sehr bekannten Lehrbuchs der „Darstellenden Geometrie“ enthält nicht weniger als 264 Figuren, 4 Tafeln, 485 kотиerte Übungsaufgaben und 89 ausgeführte Beispiele. Der dargebotene Stoff wird in 6 Kapitel gegliedert: (I) Schrägruß, (II) Normalrisse der geometrischen Grundgebilde, (III) Zentralprojektion (Perspektive), (IV) Orthogonale Axonometrie, (V) Kottierte Projektion, (VI) Kartenprojektion. Das umfangreichste dieser Kapitel ist das zweite. Nach Erledigung der Grundaufgaben werden darin insbesondere eckige Strahlenflächen, Schattenkonstruktionen (sehr ausführlich), Durchdringungen eckiger Strahlenflächen behandelt. Nicht minder ausführlich werden krumme Strahlenflächen, Kugel und Drehflächen besprochen. Die orthogonale Axonometrie enthält eingehende Abschnitte über die Darstellung von Kreisen, Kegel-, Zylinder- und Kugelflächen. Die beiden letzten Kapitel bringen bemerkenswerte praktische Anwendungen (Theorie und An-

wendung des Böschungskegels, gnomonische Projektion, stereographische Projektion mit vielen Beispielen). Zusammenfassend möchte man diesem Lehrbuch der Darstellenden Geometrie den Standard technischer Hochschulen zuerkennen, obwohl der Autor und seine Mitarbeiter sich zunächst nur an höhere Schulen wenden. Doch ist diese Bescheidenheit verständlich, wenn man den außergewöhnlich hohen Stand kennt, den Lehre und Forschung auf dem Gebiete der „Darstellenden Geometrie“ gerade in Wien mit E. Müller's vorbildlichem Wirken erreicht haben.
M. Pini (Köln).

Petrenko, A. I. Main-route perspective-conical coordinates and projections. Ukrain. Mat. Ž. 7 (1955), 142–159. (Russian)

Ansermet, A. A propos de deux invariants relatifs aux projections conformes en géodésie. Schweiz. Z. Vermessg. Kulturtech. Photogr. 53 (1955), 157–160.

Maruyama, Takaharu. On generalized development projections. J. Gakugei. Tokushima Univ. Math. 5 (1954), 49–56.

Draheim, Heinz. Allgemeine Reduktionsformeln für konforme Abbildungen und die Verwendung von Abbildungen grosser Formtreue für die Lösung der 2. Hauptaufgabe. Z. Vermessungswesen 80 (1955), 339–351.

Godeaux, Lucien. Une congruence linéaire de coniques. Mathesis 64 (1955), 337–340.

Schuster, Jan. On a projective generalization of the chordal line. Časopis Pěst. Mat. 80 (1955), 202–205. (Czech)

Verfasser untersucht, wie der Punkt $S(x_0, y_0, z_0)$ der projektiven Ebene gewählt werden muß, damit seine Polaren q_k ($k=1, 2$) bezüglich der beiden Kegelschnitte l_k :

$$Ax^2 + 2Bxy + Cy^2 + 2axyz + 2b_kxz + p_kz^2 = 0 \quad (k=1, 2)$$

diese Kegelschnitte in Punkten schneiden, die ihrerseits wieder auf Kegelschnitten eines vorgegebenen Systems Σ liegen. Die homogenen Koordinaten x, y, z sind dabei so gewählt, daß die Basis-Punkte des Systems Σ auf der Geraden $z=0$ liegen. Jede der Polaren q_k bildet mit $z=0$ einen zerfallenden Kegelschnitt und zusammen mit dem entsprechenden l_k (je) ein Kegelschnittsbündel σ_k . Die an die Schnittpunkte gestellte Forderung führt notwendig auf die Existenz eines gemeinsamen Kegelschnitts der Büschel σ_1 und σ_2 . Aus den so entstandenen Beziehungen gewinnt Verfasser als geometrischen Ort der gesuchten Punkte S eine Gerade, die Projektivchordale der Kegelschnitte l_1 und l_2 . Durch Hinzufügen eines weiteren Kegelschnittes l_3 ergibt sich im allgemeinen ein Tripel von Projektivchordalen mit einem gemeinsamen Punkt (Projektivchordalenzentrum). Im Falle von Kreisen kommt man auf bekannte Beziehungen zurück.
M. Pini (Köln).

Schuster, Seymour. Pencils of polarities in projective space. Canad. J. Math. 8 (1956), 119–144.

In den ∞^1 ebenen komplexen Polaritäten, die ein festes Polardreieck gemein haben und bei denen die Bildgerade zu einem festen vierten Punkt ein Geradenbüschel durchläuft, bilden die Ordnungskurven nach Coxeter [The real projective plane, 2nd ed., Cambridge, 1955;

MR 16, 1143] ein Kegelschnittbüschel mit vier einfachen oder zwei zweifachen Grundpunkten. Betrachtet man dagegen ∞^1 Polaritäten, die durch ein selbstpolares Fünfeck bestimmt sind, von denen drei Ecken fest sind und die beiden anderen Ecken auf zwei festen Geraden von einer dritten Geraden ausgeschnitten werden, die ein Büschel durchläuft, so können als Ordnungskurven weitere Typen von Kegelschnittbüschel auftreten: das Büschel mit vier einfachen Grundpunkten oder einem zweifachen und zwei einfachen Grundpunkten. Verf. untersucht Büschel von ∞^1 räumlichen Polaritäten, die durch ein festes Polartetraeder ABCD bestimmt sind und in dem die Bildebene zu einem festen fünften Punkt ein Ebenenbüschel durchläuft mit der Achse p' . Je nachdem 0, 1, 2 oder 3 Kanten von ABCD durch p' getroffen werden, haben diese Polaritäten besondere Eigenschaften, die Verf. genauer untersucht. Insbesondere bilden die Ordnungsfächen eines solchen Büschels von Polaritäten vier verschiedene Typen nichtentarteter Büschel von Flächen zweiter Ordnung. Die hexagonalen Büschel von räumlichen Polaritäten ergeben dagegen nur drei dieser Typen, dafür noch zwei weitere Typen; die restlichen sieben Typen können nicht auftreten. — Abschliessend überträgt Verf. seine Untersuchungen auf Büschel von Polaritäten des projektiven komplexen R_n . Die Klassifikation der entstehenden Büschel von Ordnungsfächen wird jedoch hier zu weitläufig.
R. Moufang (Frankfurt a.M.).

Rožanskaya, Yu. A. Some questions of the axiomatics of Euclidean geometry. Moskov. Oblast Pedagog. Inst. Uč. Zap. Trudy Kafedr Mat. 20 (1954), 59–123. (Russian)

The article is of an expository nature, following Hilbert, but with some deviations; e.g. the continuity axioms are used at an earlier stage, and spatial geometry is treated concurrently with plane geometry. Special emphasis is laid on the concepts of completeness and consecutiveness. The latter means obtaining the full implications of the axioms in one set (like those relating to incidence), before adding new concepts. Worth mentioning are: a detailed proof of Jordan's curve theorem for polygons on the basis of the axioms of incidence and order alone; an axiomatic discussion of affine geometry with respect to consecutiveness (where the first set of axioms consists of the incidence and parallel axioms); separating out the axioms referring to one straight line (where it turns out that they are complete in the euclidean, but not in the affine case); relating the concept of completeness to group theory so that the automorphisms of a certain group correspond to the different realizations of a set of axioms.
H. Busemann (Los Angeles, Calif.).

Lenz, Hanfried. Zur Begründung der analytischen Geometrie. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. 1954, 17–72 (1955).

This paper deals mainly with axiomatics of projective, affine and generalized Cayley geometries. In a first expository chapter, the author gives axiomatic definitions for desarguesian projective and affine geometries, where points, straight lines, incidence relation and, in the affine case, parallelism are taken as fundamental concepts. A few elementary properties (existence of a base, dimension formula, etc.) are then derived for these geometries [see R. Baer, Linear algebra and projective geometry, Academic Press, New York, 1952; MR 14, 675]. Most developments are valid for both finite- and infinite-dimensional cases. In chapter II, a new axiomatic is given

for the projective spaces; here, the hyperplane replaces the straight line as basic concept; this axiomatic proves useful in developing the theory of polarities. In the remaining chapter, absolute polarities are introduced in projective n -spaces — i.e., generalized Cayley geometries are defined — by means of orthogonality axioms; this is done in two ways according as the orthogonality relations involve lines (when only the elliptic case is considered if $n \geq 3$), or hyperplanes. In the latter case, one axiom can be slightly modified in order to define generalized euclidean geometries (i.e. affine geometries with absolute polarities at infinity).

J. L. Tits (Brussels).

Karzel, Helmut. Verallgemeinerte absolute Geometrien und Lotkerngeometrien. Arch. Mat. 6 (1955), 284–295.

Interpreting the lines of a plane geometry as the reflections in them leads to a system of axioms of plane geometry of the following type: Consider a group G whose center consists of the identity only. That the element α is involutic is indicated by $\alpha \text{ inv}$. It is assumed that G has a system E of generators (denoted by small roman letters) which are involutic and such that: I. If $a \neq b$, then $abx \text{ inv}$, $aby \text{ inv}$ imply $xyz \in E$. Ia. $abc \neq 1$ (this excludes elliptic geometry).

If $a \neq b$ then the totality of all $x \in E$ with $abx \text{ inv}$ is called the pencil P_{ab} . There is a set B of pencils, called proper pencils, such that: II₁) every a lies in at least two distinct proper pencils; II₂) if $P_{ab} \in B$ and P_{cd} is any pencil, then $P_{ab} \cap P_{cd} \neq \emptyset$. The lines $a \neq b$ are called perpendicular if $(ab)^2 = 1$. According to a previous result of the author [Arch. Math. 6 (1955), 66–76; MR 16, 395] either no line is perpendicular to itself, or every line is. In the first case the geometry is called regular, in the second a "Lotkern" geometry. Any line carries the same number of proper points (=pencils). In any finite geometry the theorems of Desargues and Pappus hold. The regular finite geometries are the singular euclidean geometries, the finite "Lotkern" geometries have coordinate fields of characteristic 2. If any (not necessarily finite) "Lotkern" geometry can be imbedded in a Desarguesian plane, then the coordinate field has characteristic 2, is commutative and possesses an irreducible separable polynomial of degree 2. Conversely, any such field yields a "Lotkern" geometry.

H. Busemann (Los Angeles, Calif.).

Barlotti, Adriano. Un'estensione del teorema di Segre-Kustaanheimo. Boll. Un. Mat. Ital. (3) 10 (1955), 498–506.

In einer endlichen ebenen Geometrie über einem Galoisfeld mit $q = p^h$ ($p \neq 2$) Elementen heisst oval jede Reihe von $q+1$ Punkten, von denen keine drei kollinear sind. Jeder nichtentartete Kegelschnitt ist ein Oval. Auch die Umkehrung gilt, falls $p \neq 2$ ist. Für den dreidimensionalen Raum beweist Verf. für nichtentartete Quadriken folgende analogen Sätze: 1) Eine Menge von q^2+1 Punkten, von denen nicht drei kollinear sind, ist eine nichtgeradlinige Quadrik. 2) Eine Menge von q^2+2q+1 Punkten derart, dass jede Gerade, die mehr als zwei von ihnen enthält, ganz zur Menge gehört, ist eine geradlinige Quadrik oder zerfällt in eine Ebene und eine Gerade in nicht vereinigter Lage. 3) Eine Menge von q^2+q+1 Punkten derart, dass jede Gerade, die mehr als zwei dieser Punkte enthält, ganz zur Menge gehört, ist eine Ebene oder ein quadratischer Kegel.

R. Moufang.

Panella, Gianfranco. Caratterizzazione delle quadriche di uno spazio (tridimensionale) lineare sopra un corpo finito. Boll. Un. Mat. Ital. (3) 10 (1955), 507–513.

In einem dreidimensionalen linearen Raum S_3 über einem endlichen Körper der Charakteristik $p \neq 2$ betrachtet Verf. Punktmengen Q mit den Eigenschaften: 1) in Q existiert ein Punkt P so, daß die allgemeine Gerade durch P mit Q genau noch einen weiteren Punkt gemein hat; 2) ausgenommen von dieser Forderung sind die Geraden durch P , die in einer Ebene, der sogenannten Tangentialebene in P an Q liegen; 3) wenn 3 Punkte von Q auf einer Geraden liegen, gehört jeder Punkt dieser Geraden zu Q . Hier heisst Q nichtgeradlinige Quadrik, wenn Q nicht drei kollineare Punkte enthält, andernfalls geradlinige Quadrik. Verf. beweist einige Sätze über solche Mengen, insbesondere gibt er eine einfache notwendige und hinreichende Bedingung dafür an, dass eine Punktmenge des oben genannten S_3 genau die Punkte einer nichtgeradlinigen bzw. geradlinigen Quadrik bilden.

R. Moufang.

Segre, Beniamino. Curve razionali normali e k -archi negli spazi finiti. Ann. Mat. Pura Appl. (4) 39 (1955), 357–379.

Über einem Galoisfeld mit q Elementen, $q = p^h$, $p \neq 2$, konstruiert Verf. einen linearen projektiven Raum S_r von r Dimensionen und nennt eine irreduzible algebraische Mannigfaltigkeit V_1 erster Dimension und r -ten Grades, die nicht in einem Unterraum kleinerer Dimension liegt, eine rationale Normalkurve des $S_{r,q}$. Falls $q > r+1$, enthält jede solche rationale Normalkurve genau $q+1$ Punkte mit der Parameterdarstellung $x_0 : x_1 : \dots : x_r = 1 : t : t^2 : \dots : t^r$ außer dem Punkt $0 : 0 : \dots : 0 : 1$. Die Anzahl dieser Kurven wird bestimmt. Ein k -Bogen des $S_{r,q}$ heisse jede Menge von k Punkten des $S_{r,q}$, die linear unabhängig sind für $k \leq r$ resp. von denen nicht $r+1$ in derselben Hyperebene liegen für $k \geq r+1$. Verf. beweist im Rahmen allgemein formulierter Probleme u.a. folgende Sätze: Für $r=2, 3, 4$ ist $k=q+1$ die grösste Zahl so, dass in $S_{r,q}$ ein k -Bogen existiert. Ist $r \geq 4$, $q \geq r+2$, so gibt es in $S_{r,q}$ keinen k -Bogen mit $k \geq q+r-2$. Für $r=2, 3$ und $q > r+1$ gilt, dass jeder $(q+1)$ -Bogen des $S_{r,q}$ eine rationale Normalkurve ist. Zu einem gegebenen Paar r, q mit $q > r+1$ ist jeder k -Bogen für $k \leq r+3$ in einer rationalen Normalkurve enthalten, wobei diese Kurven genauer charakterisiert werden. Insbesondere ist für $r=2$ jeder q -Bogen enthalten in einem Kegelschnitt, der für $q > 3$ eindeutig bestimmt ist.

R. Moufang.

Lombardo-Radice, Lucio. Su alcuni caratteri dei piani grafici. Rend. Sem. Mat. Univ. Padova 24 (1955), 312–345.

Ein genaues Studium der freien Erweiterungen S_k^n ($k=1, 2, \dots$) eines ebenen n -Punktes $S_1^n = P_1^n$ im Sinne von M. Hall [Trans. Amer. Math. Soc. 54 (1943), 229–277; MR 5, 72] gibt Verf. Anlass zur Einführung mehrerer neuer Begriffe, die durch Beispiele erläutert werden und die als Ausgangspunkt weiterer Untersuchungen des Verf. über graphische Ebenen dienen sollen. In der Folge der S_k^n wird zwischen den Punktstadien $S_{2k+1}^n = P_{k+1}^n$ und den Geradenstadien $S_{2k}^n = R_k^n$ unterschieden, die jeweils nur abgeschlossen sind bezüglich der Operation des Schneidens resp. des Verbindens. Die Vereinigungsmenge der Partialebenen S_1^n, S_2^n, \dots ist die von dem freien n -Punkt erzeugte freie Erweiterung P^n . Für den Begriff des Schließungssatzes ist grundlegend eine zusätzliche Inzidenzforderung in P^n , derzufolge die Seiten eines Dreiecks A, A', A'' , die in P^n nicht kollinear sind, zu-

sammenfallen sollen. Das entstehende Konfigurationstheorem C (Schliessungssatz; italienisch: proposizione configurazionale) wird durch das Schema aller seiner Inzidenzen beschrieben. Grad von C heisst die natürliche Zahl g , so dass A, A', A'' zu P_g^n , aber nicht alle zu P_{g-1}^n gehören. Der Rang von C ist der Rang der in P_g^n enthaltenen freien Unterebene, die aus C durch Weglassen der Kollinearität von A, A', A'' entsteht. Die Forderung der Kollinearität von A, A', A'' stellt ein im engeren Sinne projektives Theorem dar, wenn sie unabhängig ist von den vorhergehenden freien Inzidenzaussagen. Die möglichen Typen von Konfigurationstheorem vom Grad 2 und 3 für eine P^4 werden aufgestellt.

Der untere Freiheitsgrad einer zu S_1^n isomorphen Partialebene \bar{S}_1^n einer graphischen Ebene P ist die natürliche Zahl h so, dass \bar{S}_h^n isomorph ist zu S_h^n , aber nicht mehr \bar{S}_{h+1}^n isomorph zu S_{h+1}^n ist. Der $(n-3)$ -te untere Freiheitsgrad einer graphischen Ebene ist die eventuell nicht endliche Zahl $g_{n,P}$ so, dass 1) $g_{n,P}=0$, wenn in P keine zu S_1^n isomorphe Partialebene existiert, 2) $g_{n,P}=h_{n,P}$ ist, falls die in P enthaltenen zu S_1^n isomorphen Partialebenen den grössten unteren Freiheitsgrad $h_{n,P}$ haben, 3) $g_{n,P}=\infty$, wenn die Menge der unteren Freiheitsgrade der $\bar{S}_1^n CP$ nicht beschränkt ist. Die Folge $g_{4,P}, g_{5,P}, \dots$ heisst die erste Signatur von P . Als Beispiele betrachtet Verf. die endlichen Ebenen über $GF(2)$, $GF(3)$ und $GF(p)$. — Die Grösse $g_{n,P}$ ist geeignet, Fallunterscheidungen zwischen sogenannten homogenen und nicht-homogenen graphischen Ebenen zu erfassen, das sind solche, in denen Schliessungssätze universell bzw. nicht universell gelten.

Eine Partialebene C aus P_h^n heisst vollständig im P_h^n , wenn C zugleich mit jedem Element auch die beiden definierenden Elemente im sukzessiven Aufbau von P_h^n enthält. C heisst vom Grad h , wenn C mindestens einen Punkt vom Grad h , aber keinen Punkt vom höheren Grad enthält. Sei $\bar{S}_1^n = P_1^n$ in einer graphischen Ebene P enthalten und sei \bar{C} eine Partialebene von P_h^n , die einer vollständigen Partialebene CCP_h^n isomorph ist und so beschaffen ist, dass drei Punkte von \bar{C} genau dann kollinear in P sind, wenn sie es in \bar{C} sind; dann heisst \bar{C} eine freie Partialebene vom Grad h über P_1^n . Der n -te obere Freiheitsgrad von P_1^n in P ist die obere Grenze der Grade der freien Partialebenen über P_1^n . Die obere Grenze der oberen Freiheitsgrade der $P_1^n CP$ heisst der n -te obere Freiheitsgrad von P , der eventuell nicht endlich ist. Die Kette der oberen Freiheitsgrade von P für $n=4, 5, \dots$ heisst die zweite Signatur von P . In einer linearen Ebene P über einem Körper der Charakteristik $p \neq 0$ ist der obere Freiheitsgrad G jeder P_1^4 gleich $2(p-1)$. Verf. definiert nun als Charakteristik p einer graphischen Ebene die Zahl $\frac{1}{2}(G+2)$, mit der Konvention, dass für $G=\infty$ auch $p=\infty$ sein soll, was dem Fall der Charakteristik 0 in der sonst üblichen Bezeichnung entspricht. [Man vergleiche auch G. Pickert, „Projektive Ebenen“, Springer, Berlin, 1955; MR 17, 399]. R. Moufang (Frankfurt a.M.).

Wagner, A. On finite non-desarguesian planes generated by 4 points. Arch. Math. 7 (1956), 23–27.

The author gives a class of finite projective planes in which at least one quadrangle generates a non-desarguesian subplane. L. J. Paige (Los Angeles, Calif.).

★ Montel, Paul. Les débuts de la géométrie finie. Colloque sur les questions de réalité en géométrie, Liège, 1955, pp. 27–37. Georges Thone, Liège; Masson & Cie, Paris, 1956. 250 fr. belges; 1900 fr. français.

Aczél, J.; und Varga, O. Bemerkung zur Cayley-Kleinschen Massbestimmung. Publ. Math. Debrecen 4 (1955), 3–15.

An analytical proof is given for the fact that in the real hyperbolic Cayley plane, every invariant of a finite system of points $\{a_i\}$ is a function of the (a_i, a_j) , where (x, y) is the symmetric bilinear form associated with the fundamental conic [more general results are to be found, e.g., in H. Weyl, The classical groups, Princeton, 1939, ch. II; MR 1, 42]. This, and its analogue in the elliptic case, is used to show that in both planes, the usual distance, defined to within a constant factor, is, as was previously known, the only continuous positive distance function which is additive for collinear points (the latter condition should be stated more carefully in the elliptic case). J. L. Tits.

Laptev, B. L. The volume of a pyramid in a Lobachevskii space. Kazan. Gos. Univ. Uč. Zap. 114, no. 2 (1954), 53–77. (Russian)

Consider a rectangular tetrahedron $ABCD$ in the three-dimensional hyperbolic space with space constant k . We assume that BD is perpendicular to the plane ADC . Then $\nu = \angle ADC$ and $\mu = \angle BCD$ equal the dihedral angles at the edges BD and AC . Denote the dihedral angle at AB by λ . Formulas for the volume W of $ABCD$ in terms of λ, μ, ν are derived. First in the “asymptotic” case, where B moves to ∞ . Then a general rectangular tetrahedron is expressed as the algebraic sum of 4 asymptotic rectangular tetrahedra. Finally, the general formula

$$W = \frac{1}{6}k[L(\delta + \mu') + L(\delta - \mu') - 2L(\delta) + L(\delta + \nu') + L(\delta - \nu') - L(\delta + \lambda) - L(\delta - \lambda)]$$

is obtained, where $\mu' = \frac{1}{2}\pi - \mu$, $\nu' = \frac{1}{2}\pi - \nu$,

$$\cos \delta = \sin \mu' \sin \nu' (\sin^2 \mu' + \sin^2 \nu' - \sin^2 \lambda)^{-\frac{1}{2}},$$

$$L(x) = - \int_0^x \log \cos t \, dt \quad (-\frac{1}{2}\pi \leq x \leq \frac{1}{2}\pi).$$

All these results, including the use of $L(x)$, are due to Lobachevskii. The present paper is merely a modernization of his approach. H. Busemann (Los Angeles, Calif.).

Garaj, Jozef. On application of imaginary coordinates in the geometry of Minkowskian four-dimensional spacetime. Mat.-Fyz. Časopis. Slovensk. Akad. Vied 5 (1955), 114–123. (Slovak. Russian summary)

On démontre quelques résultats algébriques bien connus du calcul vectoriel dans l'espace de Minkowski en partant du système des verseurs pour lesquels

$$i_1 \cdot i_1 = i_2 \cdot i_2 = i_3 \cdot i_3 = 1, \quad i_4 \cdot i_4 = 1 \quad (i_4 = ii_4).$$

F. Vyčichlo (Prague).

See also: Sokolov, p. 704; Schützenberger, p. 704; Busemann, p. 779.

Convex Domains, Extremal Problems, Integral Geometry

Krasnosel'skii, M. A. On a proof of Helly's theorem on sets of convex bodies with common points. Voronež. Gos. Univ. Trudy. Fiz.-Mat. Sb. 33 (1954), 19–20. (Russian)

A proof of Helly's theorem is based on the fact that if an n -simplex be covered by $n+1$ closed sets so that the k th set misses the k th face of the simplex, then these sets have

a common point [Knaster, Kuratowski, and Mazurciewicz, *Fund. Math.* 14 (1929), 132-137]. *V. L. Klee.*

Helfenstein, Heinz G. Ovals with equichordal points. *J. London Math. Soc.* 31 (1956), 54-57.

A point inside a closed convex curve is called an equichordal point if every line passing through the point meets the curve in two points a constant distance apart. It is not known whether closed convex curves with two equichordal points exist. The author demonstrates analytically the non-existence of such curves subject to a further condition: the curve is to be at least six times differentiable at the two points where it is intersected by the line drawn through the two equichordal points.

G. A. Dirac (Vienna).

Fenchel, Werner. Über konvexe Funktionen mit vorgeschriebenen Niveaumannigfaltigkeiten. *Math. Z.* 63 (1956), 496-506.

Suppose C is an open convex subset of E^n and ϕ is a twice-differentiable non-constant real-valued function on C . The author obtains necessary and sufficient conditions for the existence of a twice-differentiable convex function having the same level sets as ϕ . For each $x \in C$, let P_x be the quadratic form $\sum_{i,j} (\phi_{ij}(x)) y_i y_j$, $y \in E^n$, P_x^* the restriction of P_x to the hyperplane $\{y: \sum_i (\phi_i(x)) y_i = 0\}$, r_x the rank of P_x , and S_x [resp. S_x^*] the value of the elementary symmetric function of order r_x [resp. $r_x - 1$] in the characteristic roots of P_x [resp. P_x^*]. Then there exists a strictly increasing twice-differentiable real-valued function F with $\text{dmn } F = \text{rng } \phi$ and $F \circ \phi$ convex if and only if the following three conditions are all satisfied: (a) At each of its critical points (if any), ϕ attains an absolute minimum. (b) For each $x \in C$, the form P_x^* is positive semi-definite and its rank is at least $r_x - 1$. (c) For each t interior to $\text{rng } \phi$, $\sup_{x \in C} (-S_x) / [S_x^* \sum_i (\phi_i(x))^2] = G(t) < \infty$, and the function G is majorized on $\text{int rng } \phi$ by the logarithmic derivative of a differentiable function on $\text{rng } \phi$ which is positive on $\text{int rng } \phi$.

The problem is also studied without differentiability assumptions, and it is proved (roughly speaking) that the level sets of a given convex function must all have the same directions of unboundedness. A more detailed discussion of this and related matters appears in the author's "Convex cones, sets, and functions" [from lecture notes by D. W. Blackett, Princeton, 1953], in which the theorem of the first paragraph is also proved. See also de Finetti [Ann. Mat. Pura Appl. (4) 30 (1949), 173-183; MR 13, 271]. *V. L. Klee, Jr. (Los Angeles, Calif.).*

★ **Tucker, A. W. Linear inequalities and convex polyhedral sets.** Proceedings of the Second Symposium in Linear Programming, Washington, D.C., 1955, pp. 569-602. National Bureau of Standards, Washington, D.C., 1955.

The paper investigates the structure of the set of solutions of a system of linear inequalities, $(a_h, x) \geq b_h$, $h = 1, \dots, m$, where a_h is a vector in R_n . The solution set S is called a convex polyhedral set. For every subset H of the indices $1, \dots, m$, the face S_H is defined as all vectors x such that $(a_h, x) = b_h$ for $h \in H$, $(a_h, x) > b_h$ for $h \notin H$. (Many of the faces will of course be empty.) The faces S_H form a partition of S into open convex polyhedral sets of various dimensions. Faces of dimension 0 and 1 are called vertices and edges respectively. Typical theorems: (1) For the homogeneous case $(a_h, x) \geq 0$, if the system is non-singular then the set of solutions consists of the

origin and all vectors expressible as the sum of vectors belonging to edges. (2) For the general non-singular system the set of solutions consists of the sum of the convex hull of the vertices of S plus the solutions of the associated homogeneous system.

The paper contains numerical examples with detailed illustrations. The proofs are carried out in purely algebraic fashion being valid over any ordered field.

D. Gale (Providence, R.I.).

Meschkowski, Herbert. Elementare Behandlung von Lagerungsproblemen. *Math.-Phys. Semesterber.* 4 (1955), 256-262.

The aim of the author is to find methods for solutions of questions connected with packing which are elementary and applicable with slight modifications to corresponding problems in elliptic, euclidean and hyperbolic geometries. Such a solution, with proof, is given for the covering of least density of the sphere with congruent spherical caps. It is based on the theorem, which is true in all three geometries, that of all the triangles drawn inside a circle the inscribed equilateral triangles have the greatest area. A proof of this applicable to all three cases is given. A solution of the kind required is also outlined for the problem of the densest packing of n (non-congruent) non-overlapping spherical caps on a sphere of unit radius.

G. A. Dirac (Vienna).

Fan, Ky; Taussky, Olga; and Todd, John. An algebraic proof of the isoperimetric inequality for polygons. *J. Washington Acad. Sci.* 45 (1955), 339-342.

The authors first outline the reduction of the general problem to the convex equilateral case. Then they prove that if z_1, z_2, \dots, z_n are any $n (\geq 3)$ complex numbers and $z_{n+1} = z_1$ then

$$\sum_{j=1}^n |z_j - z_{j+1}|^2 \geq 2 \tan \frac{\pi}{n} \Im \sum_{j=1}^n \bar{z}_j z_{j+1}$$

with equality if and only if $z_j = \alpha \exp(2\pi i j/n) + \beta$ ($1 \leq j \leq n$), where α and β are two arbitrary complex numbers. The proof is based on extremal properties of the characteristic values of Hermitian matrices. The isoperimetric inequality for equilateral polygons is an immediate consequence of this result if the complex numbers z_1, z_2, \dots, z_n are regarded as the vertices of an equilateral polygon.

G. A. Dirac (Vienna).

Ratray, B. A.; and Peck, J. E. L. Infinite stochastic matrices. *Trans. Roy. Soc. Canada. Sect. III.* (3) 49 (1955), 55-57.

In the space of all real infinite matrices whose rows and columns are absolutely summable, define a topology by asserting that (x_{ij}) is in a (N, ϵ) neighborhood of 0 if

$$\sum_j |x_{ij}| < \epsilon \quad (i \leq N), \quad \sum_i |x_{ij}| < \epsilon \quad (j \leq N).$$

The authors show that, in this topology, the convex closure of the permutation matrices is the set of doubly stochastic matrices. This result, like that of an earlier paper by Isbell [Proc. Amer. Math. Soc. 6 (1955), 217-218; MR 16, 893], represents a contribution to Birkhoff's Problem 111 [Lattice theory, Amer. Math. Soc. Colloq. Publ., v. 25, rev. ed., New York, 1948; MR 10, 673].

A. J. Hoffman (Washington, D.C.).

Derry, Douglas. On closed differentiable curves of order n in n -space. *Pacific J. Math.* 5 (1955), 675-686.

Let C_n denote a closed curve in real projective n -space which meets no hyperplane ($= (n-1)$ -space) in more than

n points. For each point s of C_n and each k a linear k -space is assumed to exist, the osculating k -space of C_n at s , which is the limit of the k -spaces through any $k+1$ points of C_n as these points converge to s independently of one another ($k=0, 1, \dots, n-1$). A straight line is called an l -line if each of its points lies on n osculating $(n-1)$ -spaces. Two distinct points span an l -line if and only if they lie on the osculating hyperplanes of two n -tuples of points which alternate on C_n . From a point of C_n , an l -line of C_n is projected onto an l -line of the projection C_{n-1} , and the projection of C_n from an l -line is a C_{n-2} . These results enable the author to extend a theorem by A. Kneser on C_3 's to C_n 's: The set of those hyperplanes which meet C_n exactly k times is connected ($k=n, n-2, n-4, \dots$) [Math. Ann. 31 (1888), 507-548]. A hyperplane contains l -lines if and only if it meets C_n in less than n points, and a straight line is an l -line if and only if every hyperplane through it contains less than n points of C_n .
P. Scherk (Saskatoon, Sask.).

See also: Vituškin, p. 718; Rado, p. 767; Busemann, p. 779.

Differential Geometry

*Busemann, Herbert. The geometry of geodesics. Academic Press Inc., New York, N. Y., 1955. x+422 pp. \$9.00.

In the thirteen years since the publication of the author's "Metric methods in Finsler spaces and the foundations of geometry" [Princeton, 1942, this volume will be referred to as M; MR 4, 109] enough work has been done on metric geometry to fill several other monographs. However, the present volume aims at the broader purpose of presenting a very readable and almost self-contained treatment of most of the field. The basic concepts have been refined, some proofs made more elegant, and, most important, the methods applied to so many problems that it is now evident how fruitful is this "geometric approach to qualitative problems in intrinsic differential geometry".

Chapter I, "The basic concepts", is concerned with elementary metric-space topology. Rectifiable curves are discussed, and Menger's convexity condition is introduced: for any two points x and z , there is a point y with $xy+yz=xz$ (xy denoting the distance between x and y). Two more axioms are needed before the existence of geodesics with the usual properties can be asserted: an axiom of local prolongability of segments, and a condition guaranteeing uniqueness of prolongation. G -spaces are finitely compact metric spaces which satisfy these axioms and have the convexity property. The remainder of the chapter develops the important basic properties of G -spaces. (Most of this material is in M, although the discussion of multiplicity of a geodesic is new.) Two-dimensional G -spaces are shown to be manifolds, and those which are straight (where geodesics are unique shortest connections) are characterized.

Chapter II treats Desarguesian G -spaces, i.e., G -spaces which may be imbedded in projective space so that geodesics become straight lines. To obtain straight planes with this property, a form of Desargues' Theorem is postulated; for higher dimensions the condition that any three points lie in a plane (a two-dimensional G -space, relative to the original metric) is sufficient. A Desarguesian Riemannian G -space is either euclidean, hyperbolic, or elliptic. Because of their relation to Finsler spaces, the

most important non-Riemannian Desarguesian spaces are the Minkowskian. A more elegant approach than appeared in M is given to Minkowskian geometry. The chapter ends with a discussion of another non-Riemannian case: the Hilbert geometries constructed in convex sets of affine space.

The basic tool in a treatment of perpendiculars for a straight G -space is the notion of convex spheres; for parallels, the idea of asymptote is a natural generalization. Chapter III, entitled "Perpendiculars and parallels", uses these concepts to give characterizations of elliptic geometry (dimension greater than two), spherical spaces, Minkowski spaces, and Euclidean spaces. Most of this appeared in M, but the present arrangement is much more readable. Finally, the relations are given which hold between the assumptions made in this chapter and the conjugate or focal points of the calculus of variations.

In geometry the theory of covering spaces finds some of its most elegant applications, for the covering transformations are rigid motions. Chapter IV develops this theory for G -spaces, obtaining most of the classical results on the fundamental group, fundamental domain, and on the existence of closed geodesics. Spaces whose universal coverings are straight generalize the interesting Finsler case in which there are no conjugate points. All such G -space metrizations of a two-dimensional torus are characterized. (There are many, although, by E. Hopf's Theorem, the only Riemannian one is flat.) The analogue of Poincaré's unit-circle model of hyperbolic geometry is used to establish the existence of transitive geodesics on compact G -surfaces of higher genus with straight covering spaces, provided there are no "parallels" and the geodesics have a divergence property.

Just as in Riemannian geometry, the local condition which most easily implies that the universal covering space is straight is that of non-positive curvature. Chapter V investigates "The influence of the sign of the curvature on the geodesics," using the definition of non-positive curvature which requires that the line joining the midpoints of two sides of a small geodesic triangle be at most half as long as the third side. The term "space with convex capsules" is used for the (demonstrably) weaker definition of non-positive curvature introduced by Pedersen [Mat. Tidsskr. B. 1952, 66-89; MR 14, 1015], where the capsules (loci equidistant from segments) are locally convex. Most of the known results on the behavior of geodesics in Riemannian spaces of negative curvature can be carried over, often using only the weaker hypothesis. The second half of this chapter attacks the problem of curvature of surfaces from a different quarter, by introducing measures of angles. The results, which are almost all contained in a paper of the author's [Canad. J. Math. 1 (1949), 279-296; MR 11, 56], show that many consequences of the Gauss-Bonnet theorem do not depend on differentiability, let alone the Riemannian character of the geometry.

From the point of view of the foundations of geometry, the sixth and last chapter is the most important. In short, the result left partially solved in M is now given a definitive form: If every point of a G -space has flat local bisectors, the universal covering space is euclidean, hyperbolic, or spherical. By means of this theorem, various characterizations of these elementary spaces are given which are answers to the Helmholtz-Lie problem. Finally, Wang's theorem is proved, which characterizes every compact (and odd-dimensional non-compact) space, convex in the sense of Menger, having a pairwise transitive

group of motions. (The even-dimensional, non-compact case has been announced by J. Tits.) Indicative of the care taken to make the book as self-contained as possible is the section reviewing the hermitian, quaternionic, and Cayley elliptic geometries.

The remarkable thing about this book, covering as it does so many topics of geometry in the large, is the amount of material which represents original research of the author's. The presentation is very clear; motivation is given both for the theorems and for some of the involved proofs. The list of problems, solved and unsolved, will be a stimulation to many. A real service has been done for the current revival of interest in geometry and geometrical methods.

L. W. Green (Minneapolis, Minn.).

Ostrowski, Alessandro. Un'applicazione dell'integrale di Stieltjes alla teoria elementare delle curve piane. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 18 (1955), 373-375.

In a preceding paper the author gave simple proofs for the standard results on the evolutes and involutes of plane curves, making a minimum of assumptions [Arch. Math. 6 (1955), 170-179; MR 17, 295]. In the present note similar proofs are given replacing difference quotients and their limits by Stieltjes integrals.

P. Scherk.

Cygankov, I. V. The evolute of a plane curve. Molotov. Gos. Univ. Uč. Zap. 8, no. 1 (1953), 13-14. (Russian)

Havel, Václav. On wedge-shaped surfaces. I. Časopis Pěst. Mat. 80 (1955), 51-59. (Czech)

Soit $f(x)$ la fonction définie dans $I_x = (a, b)$, $g(y)$, $h(y)$ les fonctions données dans $I_y = (c, d)$. Toutes les fonctions possèdent les dérivées continues. La surface déterminée par l'équation $z = f(x)g(y) + h(y)$ est l'objet d'étude de l'auteur. La surface contient le système des courbes, dont chaque paire sont les courbes qui correspondent bi-univoquement dans la transformation composée d'une translation et d'une affinité.

Au cas spécial on étudie les propriétés de la surface

$$z = \frac{(a-A)x^2 + b - B}{c^2} y^2 + Ax^2 + B,$$

(a, b, c, A, B sont les constantes) qui possède les paraboles dans les plans $x = x_0$ et aussi dans les plans $z = by + q$, où h, g satisfont certaines conditions. Les projections des paraboles dans le plan $z = 0$ sont les paraboles affines; l'axe de l'affinité est l'axe x . F. Vytichlo (Prague).

Frankx, E. Sur les surfaces réglées gauches. Bull. Soc. Roy. Sci. Liège 24 (1955), 296-300.

Geidel'man, R. M. On the theory of congruences of circles in a multidimensional conformal space. Dokl. Akad. Nauk SSSR (N.S.) 102 (1955), 669-672. (Russian)

L'ensemble des circonférences dépendantes des $(n-1)$ paramètres et situées dans l'espace conforme à n dimensions est la congruence des circonférences. A l'aide de la méthode du repère mobile qui est déterminé par les points S_0 et S_{n+1} et par les hypersphères S_1, \dots, S_n passant par S_0, S_{n+1} , on déduit les invariants de la congruence. L'équation fondamentale pour le mouvement du repère est

$$dS_\alpha = \omega_\alpha^\beta S_\beta \quad (\alpha, \beta = 0, 1, \dots, n+1).$$

Les points d'intersection des courbes $S_i, S_i + dS_i$ de la congruence sont les foyers. Les coordonnées curvilignes

des foyers sur la circonférence S_i sont déterminées par le système

$$2\omega_i^{n+1} + 2p\omega_i^n - p^2\omega_i^0 = 0 \quad (i = 1, 2, \dots, n-1).$$

La surface qui est engendrée par le système à un paramètre des circonférences (de la congruence) contenant les foyers (qui coupent les circonférences successives) est „la développable de la congruence”. La courbe enveloppée par les circonférences mentionnées est l'arête de rebroussement. On peut montrer que chaque courbe de la congruence possède $(2n-2)$ foyers et $(2n-2)$ développables qui passent par la courbe. La congruence possède $(2n-2)$ hypersurfaces focales et $(2n-2)$ systèmes des développables.

En prenant les hypersphères S_i , qui touchent les $(n-1)$ hypersurfaces (F_i) engendrées par les foyers $F_i = S_0 + p_i S_n - \frac{1}{2} p_i^2 S_{n+1}$, on obtient pour la congruence le système de Pfaff

$$\omega_i^n = C_i^n \omega_j^0, \quad \omega_i^{n+1} = \frac{1}{2} p_i^2 \omega_i^0 - p_i \omega_i^n \quad (i, j = 1, 2, \dots, n+1).$$

La congruence des circonférences dans l'espace à n dimensions dépend des $(2n-2)$ fonctions des $(n-1)$ variables. La surface de canal, c'est à dire l'enveloppe du système à un paramètre des sphères, est déterminée par les équations $C_i^n = 0$ ($i = 1, \dots, n-1$). La congruence qui possède un système des surfaces de canal dépend des n fonctions des $(n-1)$ variables.

L'auteur détermine aussi les congruences, dont n systèmes des développables sont les surfaces de canal avec les arêtes de rebroussement situées sur les hypersurfaces focales engendrées par les foyers $F_g = S_0 + p_g S_n - \frac{1}{2} p_g^2 S_{n+1}$ ($g = 1, \dots, n$).

Les types des congruences mentionnées sont étudiés, en particulier les congruences dont toutes $(2n-2)$ développables qui passent par la circonférence arbitraire de la congruence sont les surfaces de canal [les congruences pseudofocales; voir B. A. Rozenfeld, Mat. Sb. N.S. 23(65) (1948), 297-313; MR 10, 403.] F. Vytichlo (Prague).

Geidel'man, R. M. Conformal bending of congruences of circles. Mat. Sb. N.S. 37(79) (1955), 435-458. (Russian)

Arbitrary congruences of circles in conformal three-space, it is shown, are conformally indeformable, but there exist classes of such congruences which permit conformal deformations of the first order. Using the formalism and classification developed in a previous paper [Mat. Sb. N.S. 29(71) (1951), 313-348; MR 13, 686], the author derives the equations of deformation, and shows that there exist conformally deformable circle congruences which depend on two arbitrary functions of two variables. There are no non-trivial second-order deformations. For the cyclic systems of Ribaucour and for congruences possessing two families of canal surfaces ("congruences K ", cf. loc. cit.) necessary and sufficient conditions for conformal deformation are given. Other particular congruences are investigated, and the results applied, via the correspondence of Darboux, to deformations of two-parameter systems of lines in four-dimensional projective space.

L. W. Green (Minneapolis, Minn.).

Blaschke, Wilhelm. Einführung in die Geometrie der Waben. Birkhäuser Verlag, Basel und Stuttgart, 1955. 108 pp. DM 15.25.

The first and second chapters deal with plane 3-webs, in particular hexagonal webs (47 pp.), and with 4-webs of

surfaces, in particular octahedral webs (30 pp.), respectively. While the "Geometrie der Gewebe" [Springer, Berlin, 1938] by Blaschke and Bol studied their invariants mostly by means of differential operators, the present treatment prefers Pfaffians as the principal tool. The last two chapters contain remarks on plane 4-webs (12 pp.) and on webs of curves in space (9 pp.). This book does not claim to replace the earlier one or to offer an introduction to the more accessible parts of the theory. The geometry of webs seems to be stressed somewhat less than its analytical apparatus. But it contains an interesting selection of subject matter and in particular a readable introduction to the calculus of exterior differential forms. A few more recent results are included, among them some of Dou's work on plane 4-webs [Mem. Real Acad. Ci. Art. Barcelona 31 (1953), 133-218; MR 16, 400]. P. Scherk.

Sulikovskii, V. I. Affine classification of surfaces with an infinite number of nets of displacement. Dokl. Akad. Nauk SSSR (N.S.) 105 (1955), 430-432. (Russian)

The determination of such nets depends on the classification of pencils of conics. There are eleven cases, characterized by the basic points $ABCD$: $(AABB)$, $(AABC)$, $(ABCD)$, $(AABE)$, $(AABB)$, $(AAAA)$, $(AABAB)$ and three types $(AAAA)$, where A and \bar{A} are complex conjugate. To each of them belongs a type of surface, e.g. to the second case, with pencil $x^2 - y^2 + z^2 + \lambda xy = 0$, belongs $z = \log \cosh y - \log \cosh x$, to the fifth, with $x^2 + y^2 + \lambda xz = 0$, belongs $z = e^y \cos x$, to one of the types $(AAAA)$, with $y^2 - xz + \lambda x^2 = 0$, belongs $x^3 - 6xy + 6z = 0$ (the surface of Cayley, to another type $(AAAA)$, with $x^2 \pm y^2 + \lambda xy = 0$, belongs $z = x^2 \pm y^2$. The minimal surface of Scherk is an example of the first case.

D. J. Struik (Cambridge, Mass.).

Sulikovskii, V. I. On infinitesimal bending of a surface. Kazan. Gos. Univ. Uč. Zap. 114, no. 2 (1954), 79-87. (Russian)

This is a unified representation by tensor methods of classical results on infinitesimal bending of surfaces as found in the well known treatises of Bianchi and Darboux on differential geometry and in A. P. Norden's "Spaces with affine connections" [Gostehizdat, Moscow-Leningrad, 1950; MR 12, 441].

H. Busemann.

Villa, M.; e Muracchini, L. L'applicabilità proiettiva di due trasformazioni puntuali. Boll. Un. Mat. Ital. (3) 10 (1955), 313-327.

In this paper the author defines the projective applicability of two point transformations between two planes and introduces the projective linear element of such a transformation, which is analogous to that of a surface in ordinary space and can be expressed as

$$\Phi = (\omega_2 \bar{\Theta}_1 - \omega_1 \bar{\Theta}_2) / (\omega_2 \Omega_1 - \omega_1 \Omega_2),$$

where $\bar{\Theta}_1, \bar{\Theta}_2$ are cubic forms and Ω_1, Ω_2 are quadratic forms in Pfaffian forms ω_1, ω_2 . It is shown that a necessary and sufficient condition for two point transformations between two planes to be projectively applicable is that at corresponding points the two transformations have the same projective linear element.

C. C. Hsiung.

Švarc, A. S. A volume invariant of coverings. Dokl. Akad. Nauk SSSR (N.S.) 105 (1955), 32-34. (Russian)

If the universal covering space \tilde{R} of a compact Riemann space R is not compact, then it has the following property: There are functions $a(r)$ and $b(r)$ such that the volume

$V(r)$ of a sphere in \tilde{R} with the arbitrary center and radius r satisfies $a(r) \leq V(r) \leq b(r)$. Thus the order of growth of $V(r)$ is a topological invariant of R . It is shown here that it is determined by the first fundamental group alone: If the elements n_1, n_2, \dots of this group are enumerated in such a way that the length λ_i of a shortest curve in n_i does not decrease, then the number of λ_i with $\lambda_i \leq r$ is the above order of growth. As applications the author proves the known theorem, that the fundamental group of a compact space with negative curvature cannot be abelian, and the apparently new theorem, that the fundamental group of a compact n -dimensional space with non-positive curvature cannot be an abelian group of rank less than n .

H. Busemann (Los Angeles, Calif.).

Hartman, Philip; and Wintner, Aurel. Regular binary Pfaffians and nonparabolic partial differential equations. Rend. Circ. Mat. Palermo (2) 3 (1954), 347-362 (1955).

In a previous paper [Arch. Math. 5 (1954), 168-174; MR 16, 703], the authors showed that the solutions of class C^2 of an overdetermined system of differential equations $f^j(x, y, u, v, u_x, u_y, v_x, v_y) = 0$ ($j = 1, 2, 3$), if certain inequalities are satisfied, will belong indeed to C^{n+1} , C^n ($n \geq 2$) being the differentiability class of the functions f^j . In the present paper, using theorems about Pfaffian forms, the authors obtain further results of the same type to solve several problems in differential geometry. Specifically, they prove, first, that a surface of class C^3 without umbilical points will be of class C^3 if and only if the Gaussian and mean curvatures are of class C^1 . Secondly, they show that, if a mapping which (along with its inverse) is of class C^1 transforms a non-singular (definite or indefinite) Riemann metric with coefficients of class C^1 into a metric with coefficients of class C^1 , then the mapping must be of class C^2 . Thirdly, they prove analogous statements about the mappings which reduce a positive definite metric to the Tchebychev normal form or to the geodesic normal form.

A. Douglis.

Sen, R. N. On pairs of teleparallelisms. II. J. Indian Math. Soc. (N.S.) 19 (1955), 61-71.

In an earlier paper [same J. (N.S.) 17 (1953), 21-32; MR 14, 1123] a type of 3-dimensional metric spaces V_3 admitting a particular pair of teleparallelisms was obtained. This type of spaces is a generalization, in a certain sense, of the spaces of constant curvature in which the corresponding teleparallelisms are Clifford's right and left parallelisms. The present paper continues the study of the spaces V_3 . The teleparallelisms are defined by means of certain orthogonal ennuples and these latter may be used to define infinitesimal transformations in the space. In spaces of constant curvature, the infinitesimal transformations associated with Clifford's parallelisms generate two simply transitive groups which are commutative with each other and satisfy Killing's equations. This no longer holds for pairs of teleparallelisms in V_3 but analogous properties are obtained which are examined in detail.

A. Fialkow (Brooklyn, N.Y.).

★ **Raszevski [Raševskii], P. K.** Wstęp do rachunku tensorowego. [Introduction to tensor calculus.] Państwowe Wydawnictwo Naukowe, Warszawa, 1955. 83 pp. 8.80 zł.

Translation of the first part of Raševskii's Rimanova geometrija i tenzornyj analiz [Gostehizdat, Moscow, 1953; MR 16, 1051].

Sun, Peng-Wang. Problem of equivalence of the integral $\int F(x, y, y', y'', \dots, y^{(n)}) dx$. Acta Math. Sinica 4 (1954), 223-243. (Chinese. English summary)

The author studies the equivalence problem of the integral $\int F(x, y, y', \dots, y^{(n)}) dx$ under the group of all contact transformations in the (x, y) -plane, by applying the method of Elie Cartan. A set of $n+2$ invariant linear differential forms in the variables $x, y, y', \dots, y^{(n)}$ are associated to the problem, by requiring that their exterior derivatives satisfy equations of a certain form. These equations do not reduce to the equations given by Cartan in his study of the case $n=2$ [J. Math. Pures Appl. (9) 15 (1936), 42-69]. S. Chern (Chicago, Ill.).

See also: E. Cartan, p. 697; Derry, p. 778; de Buhr, p. 809.

Riemannian Geometry, Connections

Nash, John. The imbedding problem for Riemannian manifolds. Ann. of Math. (2) 63 (1956), 20-63.

Continuing his work on the imbedding problem, begun in his former paper on C^1 -embeddings [Ann. of Math. (2) 60 (1954), 383-396; MR 16, 515], the author turns to the study of imbeddings of Riemannian manifolds of class C^3 or higher, and establishes that every such manifold can be embedded in a C^3 -isometric way in a Euclidean space, whose dimension can be estimated. The main part of the proof rests on a perturbation theorem of the "open-mapping" variety. This may most easily be explained as follows: Let ϕ be a non-linear map of a Banach space X of functions or tensors into another such space Y . Let $\dot{y}=A(x, \dot{x})$ be the Fréchet derivative of ϕ at the point x , and suppose that the equation $\dot{y}=A(x, \dot{x})$ can be solved for \dot{x} in terms of \dot{y} in the form $\dot{x}=F(x, \dot{y})$, so that $\dot{y}=A(x, F(x, \dot{y}))$. In this case, ϕ may be regarded as having a "non-vanishing Jacobian" in a suitable abstract sense. Nash then proves that with suitable assumptions on the analytic nature of ϕ, A, F , much weaker however than continuity of F in x , the range of ϕ covers an open set. If F were known to be continuous in \dot{y} , this could be done as usual by solving the equation $x_t=F(x, \Delta y)$ for any desired variation Δy of y . In the present case, in which F acts on x as a partial differential operator of second order, Nash modifies the equation $x_t=F(x, \Delta y)$ to a system of integral equations as follows:

$$z(\theta) = z_0 + \int_{\theta_0}^{\theta} F(S_{\theta} z, M) d\theta, \quad L(\theta) = \int_{\theta_0}^{\theta} u(\theta - \theta') \{A(z, F(S_{\theta} z, M)) - M\} d\theta'.$$

Here u is a C^∞ function vanishing for $\theta < 0$ and identically one for $\theta > 1$; S_{θ} is a suitably chosen "smoothing operator" or "mollifier" such that $S_{\theta} z \rightarrow z$ as $\theta \rightarrow \infty$, and

$$M = M(L) = \frac{\partial}{\partial \theta} \{u(\theta - \theta_0)G + L\}.$$

If θ_0 is sufficiently large, these integral equations may be solved, and $\phi(z(\infty)) = \phi(z_0) + G$. Using the main perturbation theorem reported above, together with a number of devices taken from " C^1 -isometric embeddings", the author is then able to establish Theorem 2: A compact Riemannian n -manifold with a C^k positive metric has a C^k isometric imbedding in any small volume of Euclidean $\frac{1}{2}n(3n+11)$ -space, provided $3 \leq k \leq \infty$.

An especially interesting feature of the proof is that

the bound on the number of dimensions required for the imbedding is obtained by the application of "algebraic-geometry" dimensionality arguments, whose use is justified by appeal to an earlier result of the author giving an algebraic imbedding for general differential manifolds [ibid. 56 (1952), 405-421; MR 14, 403].

The paper ends with the extension of Theorem 2 to noncompact manifolds, the result being as follows: Theorem 3: Any Riemannian n -manifold with a C^k positive metric, where $3 \leq k \leq \infty$, has a C^k isometric imbedding in $\frac{1}{2}n(n+1)(3n+11)$ -dimensional euclidean space, in fact, in any small portion of this space.

The proof uses Theorem 2 and a special device for localizing the embedding problem on a non-compact manifold, using a special covering of the manifold by neighborhoods suitably defined in terms of a triangulation.

J. Schwartz (New York, N.Y.).

Kuiper, Nicolaas H. On C^1 -isometric imbeddings. I, II. Nederl. Akad. Wetensch. Proc. Ser. A. 58=Indag. Math. 17(1955), 545-556, 683-689.

These two papers give sharpenings of the results on C^1 -isometric imbeddings of Riemannian manifolds given by Nash [Ann. of Math. (2) 60 (1954), 383-396; MR 16, 515]. The main theorems are the following. Theorem 1: If a compact C^1 -Riemannian manifold of dimension n has a C^1 -imbedding in Euclidean n -space E^n , $N \geq n+k$, $k=1$; then it has a C^1 isometric imbedding in E^N . Nash proved this for $k=2$. Theorem 4: If an open C^1 -Riemannian n -manifold has a C^1 -imbedding in E^N , $N \geq n$; then it has a C^1 -isometric imbedding in E^{N+1} . As a corollary, the author shows that the n -dimensional simply connected space of constant negative curvature has a C^1 -isometric imbedding as a closed subset of E^{n+1} .

The proof is closely related to the proof given by Nash in the paper cited above. The dimensional improvement is made possible by replacing Nash's "spiraling" perturbation, which requires $k \geq 2$ in Theorem 1, by a "corrugation" which only requires $k \geq 1$. This corrugation then makes necessary a somewhat more complicated discussion of the convergence than is needed in Nash's case.

J. Schwartz (New York, N.Y.).

Wintner, Aurel. On the local embedding problems in the differential geometry of surfaces. Amer. J. Math. 77 (1955), 845-852.

A C^n -Riemannian metric, $n \geq 1$, is called regular, if it has a Gaussian curvature of class C^{n-1} . A surface, C^{n+1} -imbedded in Euclidean space, has, a consequence of the theorem egregium, a regular C^n -metric. This notion of regular Riemannian metric, originated from H. Weyl, seems to allow improved formulations and answers to problems in the local theory of surfaces, which concern with differentiability assumptions. The main purpose of this paper is to show: 1) A regular C^2 -metric, with non-vanishing Gaussian curvature, possesses C^2 -imbeddings. 2) Such an imbedding can fail to be a C^3 -imbedding, even in the neighborhood of an elliptic point. 3) If an elliptic regular C^2 -metric possesses a C^4 -imbedding, then all of its C^2 -imbeddings are C^3 -imbeddings. Corresponding statements are valid for C^n -metrics, $n \geq 2$. Several unanswered questions are discussed. S. Chern.

Wintner, Aurel. On indefinite binary Riemannian metrics. Amer. J. Math. 77 (1955), 853-867.

It is proved that a regular, binary, indefinite C^n -metric is isometric to a C^n -metric with the "non-euclidean

conformal" normal form $2\lambda(u, v)dudv$, $\lambda \neq 0$. (For definition of regularity, cf. preceding review.) This theorem is applied to obtain parametrizations of surfaces of which the parametric curves are asymptotic curves or lines of curvature. If the Gaussian curvature is negative on a C^n -imbedded surface S , then S has a C^{n-1} -parametrization in which the parametric curves are asymptotic curves. The converse of this is also true. A similar theorem is valid for the lines of curvature at a non-umbilical point. These results are applied to clarify the differentiability assumptions of several formulas in the classical theory of surfaces. *S. Chern* (Chicago, Ill.).

Singal, M. K.; and Ram Behari. Generalization of normal curvature of a curve in a Riemannian V_n . Proc. Indian Acad. Sci. Sect. A. 42 (1955), 309-316.

Consider a congruence of curves through points of a hypersurface V_n imbedded in a Riemannian space V_{n+1} . The generalized curvature of C at any point of it, relative to the curve of the congruence through that point, is defined as the negative of the resolved part along C of the covariant derivative along C of the unit tangent to the curve of the congruence through the point. If the congruence is normal to V_n , this curvature becomes the normal curvature of C . The concepts of principal curvatures, lines of curvature, conjugate directions, asymptotic lines are generalized. The usual theorems concerning them are proved in a straightforward manner. *A. Fialkow*.

Gheorghiev, Gh. Applications of the tensor calculus to some problems of Riemannian geometry. Gaz. Mat. Fiz. Ser. A. 7 (1955), 245-262. (Romanian) Expository paper.

Solodovnikov, A. S. Geodesic (projective) transformations of Riemannian spaces. Dokl. Akad. Nauk SSSR (N.S.) 105 (1955), 419-422. (Russian)

G. Fubini [Mem. Accad. Sci. Torino (2) 53 (1903), 261-313] has given a theory of transformations of Riemannian V_n by which geodesic lines are preserved. We find in the present paper two classes of such spaces discussed, left by Fubini. The first class consists of V_n with $ds^2 = ds_0^2 + \sum \sigma ds_\alpha^2$ ($\alpha=1, \dots, r$), where ds_0^2, ds_α^2 are independent metrics depending each on its x^i, x^j, ds_α^2 not one-dimensional; σ, \dots, σ being positive functions of x^i ; here the associate metric

$$(ds^\sigma)^2 = ds_0^2 + \sum \sigma (dy^{\sigma+\alpha})^2 \quad (q > 0 \text{ the dimension of } ds_0^2)$$

has constant curvature K . The second class admits only affine transformations. If here $ds^2 = ds_0^2 + \sum ds_\alpha^2$ ($\alpha=1, \dots, t$) is a decomposition into a euclidean part ds_0^2 and non-reducible non-one-dimensional ds_α^2 , then an arbitrary affine transformation acts independently in every one of the spaces ds_0^2, ds_α^2 . *D. J. Struik* (Cambridge, Mass.).

Willmore, T. J. On compact Riemannian manifolds with zero Ricci curvature. Proc. Edinburgh Math. Soc. (2) 10 (1956), 131-133.

The author proves the theorem: A compact, orientable Riemannian manifold with positive definite metric, dimension $n \geq 4$, and zero Ricci curvature is flat if the first Betti number exceeds $n-4$. The proof uses the following two lemmas on Riemannian spaces of zero Ricci curvature: (1) if the metric can be written locally in the form

$$ds^2 = \sum_{\alpha=1}^3 g_{\alpha\beta} dx^\alpha dx^\beta + \sum_{i=1}^r (dx^i)^2,$$

then the space is flat and (2) for compact, orientable manifolds the first Betti number is equal to the number of independent parallel vector fields. *W. M. Boothby*.

Chern, Shiing-shen. On curvature and characteristic classes of a Riemann manifold. Abh. Math. Sem. Univ. Hamburg 20 (1955), 117-126.

This paper deals with relations between the curvature of a compact Riemannian manifold M (whose dimension is denoted below by n) and its Pontrjagin classes. The author has previously [Topics in differential geometry, Inst. Advanced Study, Princeton, 1951] obtained exterior polynomials ψ_k in the curvature forms Ω_{ij} ($=S_{ijkl}\omega_i\omega_j$, referred to orthonormal frames) which correspond to these classes in the sense of de Rham's Theorem. Here the same forms are given alternative expressions by use of Pfaffian functions. These are applied to deduce information from special assumptions on the curvature. For example, generalizing constant curvature, if $S_{ijkl} = -K(a_{ik}a_{jl} - a_{il}a_{jk})$, then the $\psi_k = 0$ and, if $n=2s$ and K^s/a_{ij} does not change sign, then this is the sign of $\chi(M)$ (Euler-Poincaré characteristic), which vanishes only when K^s/a_{ij} is identically zero. In this line of ideas the following unpublished theorem of J. W. Milnor is proved: If M is compact, orientable and $n=4$, and if its sectional curvatures along perpendicular plane elements always have the same sign, then $\chi(M) \geq 0$. If its sectional curvature is always positive, or always negative, then $\chi(M) > 0$.

Finally it is noted that if the dual Pontrjagin class $\bar{p}_1 \neq 0$ then M cannot be differentially imbedded in a Euclidean space of dimension $n+2l-1$. Applying this to complex projective space P_m ($m=2n$) this gives the impossibility of a differentiable imbedding in real Euclidean space of dimension $3m-1$ if m is even, $3m-2$ if m is odd. *W. M. Boothby* (Evanston, Ill.).

Blum, Richard. The metric of a conformally Euclidean space referred to a subspace. Trans. Roy. Soc. Canada. Sect. III. (3) 49 (1955), 1-5.

Vranceanu [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) 11 (1930), 385-389] has shown that in a sufficiently small neighborhood of a subspace V_n , the metric of the surrounding Euclidean space E_N may be written in terms of the fundamental tensors of the first, second and third kinds $a_{ij}, b_{ijk}, c_{ijkl}$ and the torsions t_{ijk} of the subspace V_n in E_N . Two generalizations of Vranceanu's formula when the enveloping space C_N is conformally euclidean are derived by the author. These new formulas describe the metric of C_N in a sufficiently small neighborhood of a subspace V_n . In one of these, the exact formula of Vranceanu is obtained by sacrificing the skew symmetry of the analogues of the t_{ijk} ; in the other, the form of the metric differs from that of Vranceanu but the skew symmetry property is maintained. *A. Fialkow* (Brooklyn, N.Y.).

Mutō, Yosio. On n -dimensional projectively flat spaces admitting a group of affine motions G_r of order $r < n^2 - n$. Sci. Rep. Yokohama Nat. Univ. Sect. I. 1955, no. 4, 1-18.

A space A_n with the affine connection $\Gamma_{\mu\nu}^\lambda$ admits an infinitesimal affine motion if the Lie derivative $XT_{\mu\nu}^\lambda$ constructed with ξ^λ as the vector of infinitesimal transformation is zero. This leads when $\Gamma_{\mu\nu}^\lambda = \Gamma_{\nu\mu}^\lambda$ to:

$$(1) \quad \xi^\lambda_{;\mu;\nu} = -R^\lambda_{\mu\nu}\xi^\mu,$$

where the semicolon denotes covariant differentiation.

The conditions of integrability of (1) form a set of linear and homogeneous equations in ξ^A and $\xi^A_{;B} = \xi^A_B$. If there are $n^2 + n - r$ independent equations in this set, the complete group of affine motions G_r in A_n has r parameters.

In the present paper the author determines those curved and symmetric A_n ($n \geq 3$) for which $r > n^2 - n$ and which, in addition, are projectively flat. In this case $R^A_{;BCD}$ can be linearly expressed in terms of the Ricci tensor Π_{AB} , and the integrability conditions of (1) are equivalent to

$$XS_{\mu\nu} = 0; XV_{\mu\nu} = 0,$$

where $S_{\mu\nu}$ and $V_{\mu\nu}$ are the symmetric and skew symmetric part of $\Pi_{\mu\nu}$ respectively. From this follows that there are only 3 types of A_n with the required properties: 1) T1 which admits a transitive G_r with $r = n^2$ or an intransitive G_r with $r = n^2 - 1$; $S_{\mu\nu}$ is of rank one and $V_{\mu\nu} = 0$. 2) T2 admits a transitive G_r with $r = n^2 - n + 1$; $S_{\mu\nu}$ is of rank one and $V_{\mu\nu}$ is of rank two. 3) T3 admits a transitive G_r with $r = n^2 - n + 1$; $S_{\mu\nu}$ is of rank two and $V_{\mu\nu} = 0$.

Expressions for the curvature tensor are given in terms of one vector (for T1) and two linearly independent vectors (for T2 and T3) which have to satisfy certain "adjunct" equations. Finally, an existence proof for these three types and the corresponding groups is given, from which follows that the established conditions are not only necessary but also sufficient. Reference is made to Vranceanu's result that an A_n with nonvanishing projective curvature admits at most a G_r with $r = n^2 - 2n + 5$. The condition of projective flatness can therefore be dropped for $n \geq 5$.

R. Blum (Saskatoon, Sask.).

Hu, Hou-Sung. An extension of conjugate affine connections. *Acta Math. Sinica* 3 (1953), 343-357. (Chinese. English summary)

Given a symmetric pseudo-tensor $b_{i_1 \dots i_m}$ in a space X_n of dimension n , m vectors $v_{(k)}^i$, $1 \leq k \leq m$, are called m -conjugate, if they satisfy the condition

$$b_{i_1 \dots i_m} v_{(1)}^{i_1} \dots v_{(m)}^{i_m} = 0.$$

m affine connections Γ_k are called m -conjugate, if the vectors $v_{(k)}^i$ remain m -conjugate relative to $b_{i_1 \dots i_m}$, when each $v_{(k)}^i$ is parallelly displaced with respect to Γ_k . This problem is formulated analytically and studied by the method of absolute differential calculus. In particular, a system of differential conditions is found for $b_{i_1 \dots i_m}$, which can be considered as a generalization of the Codazzi equations. A consequence of this analytical treatment is the following theorem: If $b_{i_1 \dots i_m}$ and Γ_n , $1 \leq k \leq m-1$, are given, there exists an m th affine connection which is m -conjugate to the $m-1$ given ones relative to $b_{i_1 \dots i_m}$, if and only if

$$(m-1) \binom{n+m-3}{m-2} - 1 \leq \binom{n+m-1}{m}.$$

If all the affine connections are without torsion, then the condition that $m-1$ affine connections are euclidean implies that the m th connection m -conjugate to them is also euclidean ($n > 2$). The problem for $m=2$ was studied by Norden [Affinely connected spaces, Gostehizdat, Moscow-Leningrad, 1950; MR 12, 441]. S. Chern.

Flanders, Harley. Methods in affine connection theory. *Pacific J. Math.* 5 (1955), 391-431.

This is a continuation of the author's work on a symbolic method in the theory of affine connections [Trans.

Amer. Math. Soc. 75 (1953), 311-326; MR 15, 161]. Let T_q^p denote the space of C^∞ q -forms on a C^∞ manifold, with p -vectors as coefficients. An operator A on T_0^1 to T_1^1 is called additive if $A(v+w) = Av + Aw$, where v, w are vector fields. All the additive operators form a linear space \mathcal{A} over the ring of C^∞ functions. An additive operator B is called a linear transformation, if $B(fv + gw) = fB(v) + gB(w)$, where f, g are scalar functions. The space \mathcal{D} of all affine connections is a linear variety of \mathcal{A} and is a coset of the space \mathcal{L} of all linear transformations. If d is an affine connection, then d^2 , which gives the curvature, is a linear transformation and is, moreover, a derivation. It is proved that a linear transformation B can be extended to a linear mapping $B: T_q^p \rightarrow T_{q+1}^p$, just as in the case of an affine connection. These concepts allow the consideration of families of affine connections.

The above notions are applied to give an intrinsic proof of the theorem that an affine connection can be expressed uniquely as a sum of a symmetric connection and a skew-symmetric linear transformation. Among new notions introduced are the adjoint of a linear transformation and the adjoint connection of an affine connection. The author then studies the closed $2r$ -form $\xi_r = S(\Theta^r)$, which is the trace of the r th power of the curvature matrix Θ . If the manifold is compact, this defines a real cohomology class, which is the zero class if r is odd and is a linear combination of the Pontrjagin classes if r is even. It seems to the reviewer that no complete proof of this theorem has been published, although a sufficient outline of such a proof can be found in the mimeographed notes of the reviewer, Topics in differential geometry [Princeton, 1951]. Using the symbolism of this paper, the author gives a proof of a theorem of Weil to the effect that the real cohomology class determined by ξ_r is independent of the choice of the connection. The paper ends by a discussion of projective connections. S. Chern (Chicago, Ill.).

Moór, Arthur. Metrische Dualität der allgemeinen Räume. *Acta Sci. Math. Szeged* 16 (1955), 171-196.

In the present paper a general space R_n means a manifold of covariant or contravariant vector densities (x, u_i) , (x, v^i) of weight $-\rho$, resp. $+\rho$, endowed with the fundamental function $L(x, u)$ or $L(x, v)$, which is positively homogeneous of first order in u 's or v 's and differentiable at least four times with respect to its arguments as considered by R. S. Clark [Proc. London Math. Soc. (2) 53 (1951), 294-309; MR 13, 74]. It is easy to construct the metric tensor g^{ik} or g_{ik} in a general space. The present author proves the following facts. When two general spaces R_n, R_n^* are dual to each other, i.e. when there exists a one-to-one correspondence between their elements: $x^i = x^i$, and any one of

$$u_i = g^{*-}(\rho - \rho^*)/2 u_i^*, \quad v^i = g^*(\rho - \rho^*)/2 v^{i*}, \quad u_i = g^{*-}(\rho + \rho^*)/2 g^{*i} v^i,$$

such as $g_{ik} = g_{ik}^*$, then one must have either $\rho = \rho^*$ ($\rho = -\rho^*$ in the third case) or the torsion vector $A^i = 0$ and their connection parameters and curvature tensors coincide also with each other. In order to prove these facts in the third case, he uses the osculating Riemannian space which is a generalization of that in an earlier paper [Acta Math. Acad. Sci. Hungar. 5 (1954), 59-72; MR 16, 285]. Finally there is a proof of the theorem: If the equation $u_i = g^{*-} \rho g^{*i} v^i(x, v)$ has at least one solution $v^i(x, u)$, homogeneous of the first order in u 's then one can construct a dual R_n to the given R_n^* .

A. Kawaguchi (Sapporo).

See also: Busemann, p. 779; Apté, p. 787; Apté et Lichnerowicz, p. 787; Clauser, p. 795.

Complex Manifolds

Grauert, Hans. Généralisation d'un théorème de Runge et application à la théorie des espaces fibrés analytiques. C. R. Acad. Sci. Paris **242** (1956), 603-605.

Cette Note concerne les espaces analytiques holomorphiquement complets. Rappelons que la notion d'„espace analytique“ généralise celle de variété analytique complexe (complex manifold), en admettant des singularités algébroides internes; une variété analytique complexe „holomorphiquement complète“ n'est pas autre chose qu'une „variété de Stein“, et ce concept a été étendu par l'auteur [Math. Ann. **129** (1955), 233-259; MR **17**, 80] au cas plus général des espaces analytiques.

L'auteur annonce un résultat important concernant les espaces fibrés analytiques principaux dont la base R est un espace analytique holomorphiquement complet, et la fibre un groupe de Lie complexe G . Si deux tels espaces sont topologiquement isomorphes, ils sont analytiquement isomorphes; autrement dit, R et G étant donnés, la classification topologique des espaces fibrés analytiques coïncide avec la classification analytique. La démonstration s'appuie sur d'intéressants théorèmes d'approximation: soit $H(R, G)$ (resp. $C(R, G)$) l'espace des fonctions holomorphes (resp. continues) définies dans R et à valeurs dans G ; dans $H(R, G)$ on a a priori deux notions d'homotopie: l'„homotopie holomorphe“ et l'„homotopie continue“; en fait, ces deux notions sont équivalentes parce que R est holomorphiquement complet (th. 3). De plus, tout élément de $C(R, G)$ est (continûment) homotope à un élément de $H(R, G)$ (th. 4). Soient R un espace holomorphiquement complet, R un ouvert holomorphiquement complet de R , et supposons que R soit „holomorphiquement expansible“ à R : pour qu'un élément de $H(R, G)$ puisse être approché, uniformément sur tout compact de R , par des éléments de $H(R, G)$, il faut et il suffit qu'il puisse être approché, uniformément sur tout compact de R , par des éléments de $C(R, G)$. Aucune démonstration n'est donnée.

H. Cartan (Paris).

Hitotumatu, Sin. A note on Levi's conjecture. Comment. Math. Univ. St. Paul. **4** (1955), 105-108.

Die vorliegende Note handelt über die bekannte Levische Vermutung, dass jedes pseudokonvexe Gebiet des n -dimensionalen komplexen Zahlenraumes C^n Holomorphie-(=Regularitäts-)gebiet ist. Der Verf. zeigt mit elementaren Mitteln, dass dieses Problem zu folgender Aussage äquivalent ist: In jedem schlichten, endlichblättrigen, pseudokonvexen Gebiet des C^n ist das additive erste Cousinsche Problem lösbar.

Die Richtigkeit der Levischen Vermutung ist 1953 gezeigt worden [K. Oka, Jap. J. Math. **23** (1953), 97-155; MR **17**, 82; H. Bremermann, Math. Ann. **128** (1954), 63-91; MR **17**, 81; F. Norguet, Bull. Soc. Math. France **82** (1954), 137-159; MR **17**, 81]. Alle gefundenen Beweise benutzen jedoch sehr tiefliegende Resultate der Funktionentheorie mehrerer Veränderlichen. Der Verf. glaubt, dass die obige Aussage zu einem weiteren evtl. leichteren Beweis führen könnte. Eine dazu dienliche Vermutung ist angegeben.

H. Grauert (Münster).

Behnke, Heinrich. Die analytischen Gebilde von holomorphen Funktionen mehrerer Veränderlichen. Zusammenfassender Bericht. Arch. Math. **6** (1955), 353-368.

This article, a lecture given by the author before the Mathematischen Forschungsinstitut in Oberwolfach on March 7, 1955, surveys developments since 1932.

First, developments are described which led to the theorem that holomorphy convexity (without the separation axiom) of an unbranched Riemann domain over C^n is equivalent to the domain being a holomorphy domain. Next, Runge's theorem (generalized), Cousin's problems I and II, and analytic subvarieties are discussed for holomorphy domains of C^n . The sufficiency of Levi's local conditions for the boundary (in order that a domain be a holomorphy domain) is mentioned (Oka, Bremermann). The remainder of the paper is concerned with Stein manifolds („holomorph vollständige Mannigfaltigkeiten“), with Grauert's new definition of Stein manifolds (in which the assumption of a countable base for the open sets is made unnecessary), and with Stein spaces which Grauert has shown are the holomorphically-convex holomorphy-domains over C^n („analytische Gebilde“ over C^n of holomorphic functions). D. C. Spencer (Princeton, N.J.).

Bergman, Stefan. Bounds for analytic functions in domains with a distinguished boundary surface. Math. Z. **63** (1955), 173-194.

Es sei \mathfrak{M} ein Gebiet des (komplex) zweidimensionalen komplexen Zahlenraumes C^2 , das von endlich vielen Stücken analytischer Hyperflächen berandet wird. Der Verf. versteht unter einer ausgezeichneten Randfläche \mathfrak{D}^2 von \mathfrak{M} eine minimale 2-dimensionale Menge auf dem Rande m^3 von \mathfrak{M} , auf der jede in der abgeschlossenen Hülle $\overline{\mathfrak{M}}$ vom \mathfrak{M} holomorphe Funktion ihr Maximum annimmt. Unter gewissen einschränkenden Voraussetzungen gilt für die Menge \mathfrak{F} der in $\overline{\mathfrak{M}}$ holomorphen Funktionen f eine Integraldarstellung, bei der über \mathfrak{D}^2 integriert wird. Der Verf. benutzt dieses Integral, um gewisse Sätze über Beziehungen zwischen der Werteverteilung von f auf \mathfrak{D}^2 und der Werteverteilung in \mathfrak{M} herzuleiten. Durch ein Integral über \mathfrak{D}^2 wird eine Orthonormalisierungsvorschrift in \mathfrak{F} gegeben. Die zu einem Orthonormalsystem gehörende Kernfunktion wird untersucht. Es wird u.a. gezeigt, dass sie in ganz \mathfrak{M} existiert. Schliesslich wird ein spezielles vollständiges Orthonormalsystem angegeben und eine Anwendung auf meromorphe Funktionen zweier Veränderlichen diskutiert. Im letzten Abschnitt seiner Arbeit gibt der Verf. unter einschränkenden Voraussetzungen für die Werte, die gewisse Funktionen $f \in \mathfrak{F}$ auf zweidimensionalen analytischen Flächen $\mathfrak{M} \subset \mathfrak{M}$ annehmen, Schranken an und kommt zu interessanten Ergebnissen. H. Grauert (Münster).

Martinelli, Enzo. Contributi alla teoria dei residui per le funzioni di due variabili complesse. Ann. Mat. Pura Appl. (4) **39** (1955), 335-343.

Let $f(x, y)$, $g(x, y)$ be functions, analytic in a domain Ω in R_4 . Let F and G be the analytic manifolds defined by $f=0$ and by $g=0$. It is assumed that $F \cap G$ consists of a finite number of points P_1, \dots, P_n . The multiplicity of the common zero P_j is denoted ν_j . A small 3-sphere S_3 with center P_j intersects F in a finite number of closed curves L_1, \dots, L_r , and similarly G in M_1, \dots, M_s . The local separator subgroup of F and G with respect to P_j is defined as the subgroup of the two-dimensional Betti

group of $S_3 - UL - UM$, generated by all 2-cycles C_2 in $S_3 - UL - UM$, homologous to zero in $S_3 - UL$, and also in $S_3 - UM$. If every curve L , is enclosed in a narrow torus σ , in S_3 and every M , in a narrow torus τ , in S_3 , every C_2 has a representation of the form $a(\sigma_1, \dots, \sigma_r) + b(\tau_1, \dots, \tau_r)$, where $N_s = a - b$ is uniquely determined. The separator subgroup \mathcal{G} of F and G is the subgroup of the 2-dimensional Betti group of $\Omega - F - G$ generated by the local separator subgroups of F and G with respect to P_1, \dots, P_n . Every cycle C_2 in \mathcal{G} has a unique representation in terms of the generators, and hence a unique set N_1, \dots, N_n of the corresponding values of $a - b$. Let $\varphi(x, y)$ be analytic in Ω . The author then derives the following generalization of the residue formula:

$$\frac{1}{(2\pi i)^2} \int_{C_2} \varphi(x, y) \frac{\partial(f, g)}{\partial(x, y)} dx = \sum N_j \varphi(P_j),$$

where dc is the surface element on C_2 . A formula of this type holds only for the cycles C_2 of the separator subgroup of F and G . *H. Tornehave (Virum).*

Hua, Loo-Keng. On the theory of functions of several complex variables. II. A complex ortho-normal system in the hyperbolic space of Lie-hypersphere. *Acta Math. Sinica* 5 (1955), 1-25. (Chinese. English summary)

This is Part II of a series of papers which have as main purpose the explicit determination of a complete orthonormal system of functions in the irreducible bounded symmetric domains of E. Cartan, with the exception of two exceptional ones [J. Chinese Math. Soc. 2 (1953), 288-323; MR 17, 191; cf. also the following review]. The results have been announced, as were those of Part I [Dokl. Akad. Nauk SSSR (N.S.) 93 (1953), 775-777, 983-984; MR 15, 617]. This paper gives the details of the computations which lead to the results of the second Doklady note. *S. Chern (Chicago, Ill.).*

Hua, Lo-Kên. On the theory of functions of several complex variables. III. On a complete orthonormal system in the hyperbolic space of symmetric and skew-symmetric matrices. *Acta Math. Sinica* 5 (1955), 205-242. (Chinese. Russian summary)

[Cf. the preceding review]. In this paper the author determines explicitly the complete orthonormal systems of functions in the hyperbolic spaces of symmetric and skew-symmetric matrices. These are the spaces of all complex-valued symmetric (respectively skew-symmetric) matrices Z satisfying the condition $I - \bar{Z}'Z > 0$ (types II and IV of E. Cartan). In the case of symmetric matrices $S = (s_{ij})$ consider the representation on the forms of degree f in s_{ij} induced by the transformation $S \rightarrow USU'$, where U runs over the complex general linear group. The representation space splits into irreducible subspaces, each spanned by forms of degree f in s_{ij} , which can be denoted by $\varphi_{f_1, \dots, f_n}^{(n)}(S)$, $1 \leq i \leq N(2/f_1, \dots, 2/f_n)$, where n is the order of S and f_i are positive integers satisfying $f_1 \geq \dots \geq f_{n-1} \geq f_n \geq 0$, $f_1 + f_2 + \dots + f_n = f$. It is proved that all these functions form an orthonormal system. From this the following Cauchy formula is derived:

$$f(Z) = c \int_S \dots \int_S (\det(I - Z\bar{S}))^{-1(n+1)/2} f(S) dS,$$

where

$$c = (\pi^{1/2})^{-1(n+1)/2} \Gamma(1) \dots \Gamma(n) / \Gamma(\frac{1}{2}) \dots \Gamma(\frac{1}{2}n).$$

Similar results are obtained for the case of skew-symmetric matrices.

All the papers of the author in this series are characterized by their explicitness. As a consequence several analytical results are derived to carry through the manipulations. We mention as an example the following auxiliary identity proved in this paper:

$$\sum_{i_1, \dots, i_n} \frac{\delta_{i_1, \dots, i_n}^{1, \dots, n} x_{i_1}^{n-1} x_{i_2}^{n-2} \dots x_{i_{n-1}}^1}{(1-x_{i_1}^2)(1-x_{i_2}^2 x_{i_3}^2) \dots (1-x_{i_{n-2}}^2 \dots x_{i_n}^2)} = \prod_{1 \leq i < j \leq n} (x_i - x_j) / \prod_{1 \leq i < j \leq n} (1 - x_i x_j),$$

where x_1, \dots, x_n are n variables, $\delta_{i_1, \dots, i_n}^{1, \dots, n} = 1$ or -1 according as i_1, \dots, i_n form an even or odd permutation of $1, \dots, n$, and the summation is over all the permutations i_1, \dots, i_n of $1, \dots, n$. *S. Chern (Chicago, Ill.).*

Hua, Lo-Kên. On the theory of functions of several complex variables. III. On a complete orthonormal system in the hyperbolic space of symmetric and skew-symmetric matrices. *Dokl. Akad. Nauk SSSR (N.S.)* 101 (1955), 29-30. (Russian)

Announcement of results in the paper reviewed above.

Hua, Loo-Keng. On the Riemann curvature of the non-Euclidean space of several complex variables. *Acta Math. Sinica* 4 (1954), 143-170. (Chinese. English summary)

Let D be a domain in a space of n complex variables $(z) = (z^1, \dots, z^n)$. Let $\varphi_0(z), \dots, \varphi_n(z), \dots$ be a finite or infinite sequence of analytic functions in D , such that they have no common zero in D . Suppose that $K(z, \bar{z}) = \sum_{\alpha=0}^{\infty} \varphi_{\alpha}(z) \overline{\varphi_{\alpha}(z)}$ is uniformly convergent in any compact subdomain D^* of D . Generalizing the definition of Bergmann's metric, the author considers the Hermitian metric $ds^2 = \sum_{1 \leq i, j \leq n} \partial_i \partial_{\bar{j}} \log K(z, \bar{z}) dz^i d\bar{z}^j$ and studies its curvature. This Hermitian metric is positive semi-definite. The author gives necessary and sufficient conditions such that it is positive definite. One of the conditions says that there should be n functionally independent functions among $\varphi_{\alpha}(z)/\varphi_0(z)$, $\alpha = 1, 2, \dots$. The main result of the paper, for which the existence of a positive definite Hermitian metric is essential, says that the Riemannian curvature is ≤ 2 . It is proved in two steps: 1) The Riemannian curvature R^* of the conformally equivalent metric $K^2 ds^2$ is related to that of the original curvature R by the relation $K^2 R^* = R - 2$; 2) $-R^*$ can be written as a sum of squares. The author also computes the Ricci curvature and shows by an example that the relation $R \leq 2$ cannot be improved. Under a further condition on the sequence of functions $\varphi_{\alpha}(z)$ it is shown that the Riemannian curvature R has a lower bound $-n$.

S. Chern (Chicago, Ill.).

Vesentini, Edoardo. Campi di vettori dotati di peso sopra una varietà complessa compatta. *Rend. Mat. e Appl.* (5) 14 (1955), 564-580.

This paper gives some relations between the characteristic classes and obstructions of vector and direction fields on a compact complex manifold. A vector field of type (w, \bar{w}) on a complex manifold of dimension m is defined by the components v^{α} , $1 \leq \alpha \leq m$, in each coordinate system z^1, \dots, z^m , such that, under a change of the coordinate system, they follow the transformation law

$$v^{\alpha'} = \left| \frac{\partial z}{\partial z'} \right|^{|\alpha|} \frac{\partial \bar{z}}{\partial \bar{z}'} \bar{v}^{\alpha'} \frac{\partial z^{\alpha'}}{\partial z^{\alpha}}, \quad 1 \leq \alpha, \alpha' \leq m,$$

where $|\partial z / \partial z'|$ and $|\partial \bar{z} / \partial \bar{z}'|$ denote respectively the functional determinant and its complex conjugate. If a

tangent direction denotes the class of all non-zero vectors differing from each other by a factor, one gets from the tangent vector bundle the tangent direction bundle, whose fiber is the complex projective space P_{m-1} of dimension $m-1$. From a non-singular vector field there is a derived direction field. Suppose M^r be the r -dimensional skeleton of the manifold M in a certain cellular decomposition. Suppose $m > 1$. Since P_{m-1} has the homotopy groups $\pi_i(P_{m-1}) = 0$, $3 \leq i \leq 2m-2$, $\pi_{2m-1}(P_{m-1}) \cong \mathbb{Z}$ (=additive group of integers), a continuous non-singular vector fields F over M^3 has a continuous derived direction field which can be extended over M^{2m-1} and has the secondary obstruction c_1^* [the secondary obstruction of E. G. Kundert, Ann. of Math. (2) 54 (1951), 215-246; MR 13, 374]. On the other hand, let G be a C^1 non-singular vector field over M^{2m-1} of weight $(0, 0)$. The fields F and G have a difference cohomology class $d(F, G) \in H^2(M, \pi_2(P_{m-1})) \cong H^2(M, \mathbb{Z})$. The following formula is proved:

$$d(F, G) = (\bar{w} - w)c_m,$$

where c_m is the 2-dimensional characteristic class of M . This, combined with Kundert's formula

$$c_1^* = d^m(F, G) + \sum_{i=1}^m c_i d^{i-1}(F, G),$$

where c_i , $1 \leq i \leq m$, are the characteristic classes of M and multiplication and powers are in the sense of the cup product, gives

$$c_1^* = (\bar{w} - w)^{m-1}(\bar{w} - w - 1)c_m + \sum_{i=1}^{m-1} ((\bar{w} - w)c_m)^{i-1}c_i.$$

This last formula and Kundert's formula are generalized to r vector fields. Applications are made to the cases where the vector fields are exterior differential forms of types $(1, m)$ and $(m-1, 0)$ respectively. Finally, the author considers the case when M is an irreducible non-singular algebraic variety imbedded in complex projective space and when there are on M r closed differential forms of type $(m-1, 0)$. If the locus of points at which the matrix of the coefficients of the r forms has rank $< r$ is a pure algebraic sub-variety of dimension equal to $r-1$, the homology class it carries is dual to the cohomology class $(-1)^{m-r+1}c_{m-r}$, where the latter is a generalization of c_1^* to r fields of contravariant vectors.

The author's notation of the characteristic classes differs from the current one, of which the latter has the advantage that the subscript is the complex dimension of the class.
S. Chern (Chicago, Ill.).

Goldberg, S. I. Tensorfields and curvature in Hermitian manifolds with torsion. Ann. of Math. (2) 63 (1956), 64-76.

The author considers complex analytic manifolds with a Hermitian metric $ds^2 = 2g_{\alpha\bar{\beta}}dz^\alpha d\bar{z}^\beta$ and gives generalizations of various theorems which have been found, especially by S. Bochner [Yano and Bochner, Curvature and Betti numbers, Princeton, 1953; MR 15, 989], to hold when the metric is Kähler, i.e. when the torsion tensor $S^\alpha_{\beta\gamma}$ vanishes. First, possible generalizations of constant curvature are dealt with; two definitions are given: (i) $E_{\alpha\bar{\beta}\gamma\delta} = Kg_{\gamma\delta}g_{\alpha\bar{\beta}}$ and (ii) $E_{\alpha\bar{\beta}\gamma\delta} = \frac{1}{2}K(g_{\alpha\bar{\beta}}g_{\gamma\delta} + g_{\alpha\bar{\gamma}}g_{\beta\bar{\delta}})$, $E_{\alpha\bar{\beta}\gamma\delta}$ denoting the curvature tensor, K a scalar function. Unlike the Riemannian case, $K = \text{constant}$ does not follow, nor is it assumed. In fact, in either of these cases if K is a non-zero constant, the metric is a fortiori Kähler. Consideration is given throughout to the

influence of each of the following two assumptions: (A) $S^\alpha_{\alpha\beta} = 0$ and (B) $S^\alpha_{\beta\gamma}$ analytic. Under the further assumption of compactness pseudo-harmonic and pseudo-Killing vectors and tensors are considered. For example it is shown that if (A) holds then a pseudo-Killing vector field is a Killing vector field and conversely. Special attention is given to the example of complex paralisable manifolds, and Hermitian Einstein spaces are briefly treated.
W. M. Boothby (Evanston, Ill.).

Apte, Madhumalati. Sur les isométries des variétés presque kählériennes. C. R. Acad. Sci. Paris 242 (1956), 63-65.

For the case of a pseudo-Kähler manifold V_{2n} the author sketches proofs of the following theorems: (1) If V_{2n} is compact then the maximal (connected) group of isometries of V_{2n} leaves the pseudo-Kähler structure invariant. Moreover, this still is true for non-compact V_{2n} if (i) the form $\psi = (-2\pi)^{-1}i\hat{\Omega}_\alpha^{\alpha\bar{\beta}}$ vanishes and (ii) the holonomy group \hat{H}_x is irreducible. $\hat{\Omega}_\alpha^{\alpha\bar{\beta}}$ denotes the curvature forms and $\hat{\cdot}$ is used to distinguish quantities defined relative to the pseudo-Kähler connection. (2) Let ξ denote the linear differential form defined (using the metric) by an infinitesimal transformation and let J denote the operator defining the almost complex structure. Then if V_{2n} is compact $I(\xi) = \int_{V_{2n}} \hat{R}_{\alpha\bar{\beta}\gamma\delta} \xi^\alpha \bar{\xi}^\beta \xi^\gamma \bar{\xi}^\delta d\tau$ is non-negative for every infinitesimal isometry. $I(\xi) = 0$ for an isometry implies that $J\xi$ is harmonic. Special attention is called to the case that $\hat{R}_{\alpha\bar{\beta}\gamma\delta}$ is everywhere negative definite, or everywhere zero.
W. M. Boothby.

Apte, Madhumalati; et Lichnerowicz, André. Sur les transformations affines d'une variété presque hermitienne compacte. C. R. Acad. Sci. Paris 242 (1956), 337-339.

Let V_{2p} be an almost complex Hermitian manifold, and let an automorphism mean a transformation of V_{2p} preserving both structures. It is then shown that the maximal, connected group of transformations of V_{2p} preserving the affine connection of the underlying Riemannian metric coincides with the maximal, connected group of automorphisms.

If V_m is a compact, orientable manifold with a Euclidean connection whose homogeneous holonomy group is irreducible, then the maximal connected group of affine transformations preserves the metric. As a corollary one obtains a similar theorem to the first for transformations preserving the connection of the almost complex Hermitian structure when the holonomy group of that connection is irreducible.
W. M. Boothby (Evanston, Ill.).

Algebraic Geometry

Engel, Wolfgang. Ein Satz über ganze Cremona-Transformationen der Ebene. Math. Ann. 130 (1955), 11-19.

The following theorem is proved: if P, Q are polynomials in two letters with complex coefficients whose functional determinant is 1, then the formulas $x' = P(x, y)$, $y' = Q(x, y)$ define a birational transformation of the plane into itself, whose inverse is given by formulas $x = P'(x', y')$, $y = Q'(x', y')$, where P', Q' are still polynomials. The method consists in proving that, for fixed a, b , the curves $P=a, Q=b$ have only one point of intersection at finite distance. These curves are considered as curves in the product of two projective lines. The author computes the Zeuthen-Segre invariant Z of the pencil

$P = \text{const.}$ by making use of the formulas established by Jung [Einführung in die Theorie der algebraischen Funktionen zweier Veränderlicher, Akademie-Verlag, Berlin, 1951; MR 13, 680] which determine Z in terms of local data; from the fact that Z must be 0, it follows that the curve $P=a$ has only one branch at (∞, ∞) and that the contribution of this point to the divisor of singularities is $2(k-1)^2$, k being the degree of P ; the curve $P=a$ is furthermore of genus 0. Then the multiplicity of (∞, ∞) in the intersection of $P=a$ and $Q=b$ is computed and is found to be one unit less than the total number of intersection points of these two curves, which gives the desired result.

C. Chevalley (Paris).

Gauthier, Luc. Footnote to a footnote of André Weil. Univ. e Politec. Torino. Rend. Sem. Mat. 14 (1954-55), 325-328.

Soient F et G deux formes quadratiques à $2k+2$ variables, u et v deux indéterminées. On pose $H(u, v) = \det(uF + vG)$. Etude des relations existant entre les sous variétés linéaires de dimension $k-1$ de la variété $F=G=0$, et la courbe hyperelliptique $y^2=H(x, 1)$. P. Samuel.

Samuel, Pierre. Algébricité de certains points singuliers algébroides. J. Math. Pures Appl. (9) 35 (1956), 1-6.

Soit O un point d'une variété algébroides V . L'auteur donne la définition suivante: V est analytiquement équivalente en O à une variété algébrique s'il existe un point d'une variété algébrique tel que le complété de son anneau local soit isomorphe à l'anneau local de O sur V (appelé le complété d'un anneau local algébrique). Avec ceci il est immédiat que V est analytiquement équivalente, en tout point simple de V , à une variété algébrique. Le théorème principal de ce mémoire donne la solution au problème de déterminer l'algébricité de points algébroides dans le cas des points singuliers isolés d'hypersurfaces: Si O est un point singulier isolé d'une hypersurface algébroides, alors V est analytiquement équivalente en O à une hypersurface algébrique. Pour qu'il soit possible de réduire dans certains cas la démonstration de l'algébricité

d'un point singulier algébroides au cas d'une hypersurface, l'auteur démontre le résultat suivant: Soient $A = K[[x_1, \dots, x_n]]$, $B = K[[y_1, \dots, y_r]]$ deux anneaux locaux complets sans diviseurs de zéro et avec même corps des fractions, tels que A soit un B -module de type fini et le conducteur de A dans sa clôture intégrale soit un idéal ouvert de A . Alors si B est le complété d'un anneau local algébrique, A l'est aussi. En utilisant ce résultat, on montre que si O est un point singulier d'une courbe algébroides C , alors C est analytiquement équivalente en O à une courbe algébrique.

E. Lluis (México, D. F.).

Gaeta, Federico. Sopra un aspetto proiettivamente invariante del metodo di eliminazione di Kronecker e sulle forme puntuali associate alle varietà algebriche. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 18 (1955), 148-150.

Given a set $S = \{f_i(x)\}$ of forms in the homogeneous coordinates of a projective space S_n and independent generic points y_1, y_2, \dots of S_n , one defines a sequence $S^{(0)}, S^{(1)}, \dots$ of sets of forms in x, y_1, y_2, \dots as follows: $S^{(0)} = S$, and if D_i is the g.c.d. of the forms of $S^{(0)}$, then $S^{(i+1)}$ is a set of equations for the cone with vertex y_{i+1} projecting the algebraic set determined by the set of equations $(1/D_i)S^{(0)}$, the set $S^{(i+1)}$ being found by the method of elimination of Kronecker. Stated result: The various D_i 's obtained are power products of the associated forms of the components (possibly embedded) of the algebraic set determined by S [the associated forms being those found by the method of generic projection, cf. Samuel, Méthodes d'algèbre abstraite en géométrie algébrique, Springer, Berlin, 1955, p. 44; MR 17, 300] and the exponent of each associated form in D_i is the multiplicity of the corresponding component in the "variety base" [cf. Severi, Ann. Mat. Pura Appl. (4) 26 (1947), 221-270 = Memorie scelte, Zuffi, Bologna, 1950, pp. 327-390; MR 10, 321; 12, 383] determined by S . M. Rosenlicht.

See also: Segre, p. 776.

NUMERICAL ANALYSIS

★ Hildebrand, F. B. Introduction to numerical analysis. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1956. x+511 pp. \$8.50.

The most informative review of this book would probably consist solely of its preface. Briefly, this states that the efficient use of high-speed calculating machines depends, ultimately, on the proper teaching of the operating staff. A teaching course would consist of an introduction, giving a fairly substantial grounding in the basic operations of computation, approximation, interpolation, numerical differentiation and integration, the numerical solution of equations, and the solution of ordinary differential equations. The course should not only exhibit techniques, but should also derive the formulae so that the basic hypotheses are evident and modifications apparent, cover such problems as error analysis, convergence and stability, and be reinforced by actual computation on desk machines. The way is then clear for an advanced course to cover linear algebra, partial differential equations, and integral equations, after which the student should be in a favourable position for using high-speed equipment and for modifying and developing methods appropriate to such machines.

This is worth saying and well said, and the book, based

on a course of lectures at the Massachusetts Institute of Technology, fulfills quite admirably the first part of this programme.

An introductory chapter discusses various types of approximation, and gives error analysis for some simple basic operations. In Chapter 2 we are introduced to divided differences, the fundamental interpolation formula of Newton, and the Aitken iterative method for direct and inverse interpolation. Chapter 3 obtains Lagrangian formulae for interpolation, differentiation and integration, at pivotal and non-pivotal points, and the Newton-Cotes integration formulae. In Chapter 4 the interval of tabulation is constant, and ordinary differences are defined (with the best notation) and used to produce the various interpolation formulae of Newton, Gauss, Stirling, Bessel and Everett. Difference operators are defined and used in Chapter 5 to derive finite-difference formulae for derivatives, integrals, subtabulation and the summation of series. Chapter 6 gives a selection of methods for solving ordinary differential equations, mainly but not exclusively with one-point boundary conditions. Chapter 7 considers the use of least-squares polynomial approximations for curve-fitting and smoothing, over continuous and discrete ranges, introducing the orthogonal poly-

nomials of Legendre, Laguerre, Hermite, Chebyshev and Gram, and the factorial functions. The orthogonal polynomials are used again in Chapter 8 to derive quadrature formulae of Gauss type, though purely algebraic methods are also given for this purpose. Chapter 9 considers approximation based on trigonometric and exponential series, the use of Chebyshev polynomials for economisation of power series, and continued fractions and reciprocal differences. The last chapter is concerned partly with the solution of simultaneous linear algebraic equations, with particular reference to the reduction methods of Gauss and Crout and a comment on iterative methods and relaxation, and partly with the determination of the zeros of non-linear equations and in particular of polynomials through the methods of Newton, Bernoulli, Graeffe, Lin and Bairstow.

Theory and illustrated practice go hand in hand, though the former has the stronger grip. Thus, while few formulae are ever written without error terms, the correct uses and comparisons of allied formulae and methods are usually explained, either in the text or in the problems, of which there are over 500 and which are designed both for practice and for extending the theory.

Significant omissions are rare, usually deliberate, and mostly covered by the "supplementary references", contained in a bibliography of 276 entries. It might have been desirable, however, to mention explicitly Comrie's "end-figure" method of subtabulation; the use of simultaneous "throw-back" formulae for interpolation with printed tables; the practical application of first and second sums, in difference tables, for evaluating integrals and solving differential equations; and the Aitken extension of the Bernoulli method for finding all the zeros of a polynomial.

The casual computer, looking for a method without "frills", might find this book difficult, though a Directory of Methods, reinforcing the usual index, is a useful signpost. There is certainly much space occupied with the derivation of error terms and with error analysis, and the chapter on differential equations, in particular, has somewhat involved sections on error and error propagation which tend to obscure the basic techniques. The serious student, however, for whom the book was written and who, in this reviewer's opinion, is alone worthy of serious attention, has here a splendid introduction to the art and science of computation. Great care has clearly been taken with its compilation, the errors are likely to be few, and one hopes that the author is even now hard at work on the "advanced" course. *L. Fox* (Teddington).

Scarborough, James B. *Numerical mathematical analysis*. 3rd ed. The Johns Hopkins Press, Baltimore; Oxford University Press, London, 1955. xix+554 pp. \$6.00.

This is the third edition of a much-used textbook on numerical analysis, whose first edition appeared in 1930. The book remains a good place for the new desk-computing man to learn the fundamental operations of his trade.

Besides correcting known errors in the second edition [1950; MR 12, 537], the author has added a 44-page last chapter on matrices and the solution of linear algebraic systems. Here the following topics are discussed in the author's straightforward style: evaluation of determinants; solution of linear systems by row elimination and by Seidel's single-step iteration; elementary matrix operations; matrix inversion by row elimination. There

are several detailed illustrative numerical examples and eight exercises in this chapter.

Errors noted: In formula (2), p. 519, the transpose of A^{-1} is given, not A^{-1} . In section 165 the stated conditions are sufficient for the convergence of the iteration, but are far from necessary. The latter is a restatement for linear equations of a corresponding misconception for non-linear equations found on p. 210 (2nd and 3rd editions) and p. 195 (1st ed.). *G. E. Forsythe*.

Nicholson, S. C.; and Jeanel, J. *Some comments on a NORC computation of π* . Math. Tables Aids Comput. 9 (1955), 162-164.

The authors discuss briefly and give the results of a calculation of π to 3089 decimal places by the NORC. This demonstration routine takes approximately 13 minutes and uses 2214 terms of the series

$$\pi = 4 \sum_{n=0}^{\infty} (-1)^n \{100(.2)^{2n+3} - (239)^{-1-2n}\} (2n+1)^{-1}$$

which follows from Machin's formula

$$\pi = 16 \operatorname{arccot} 5 - 4 \operatorname{arccot} 239.$$

Thus the fact that the NORC is a decimal machine is being used to advantage. The results obtained agree with the 2035D value obtained by Reitwiesner on the ENIAC [same journal 4 (1950), 11-15; MR 12, 286]. A frequency count of the various digits is given. *D. H. Lehmer*.

Markowitz, Harry. *Concepts and computing procedures for certain X_{ij} programming problems*. Proceedings of the Second Symposium in Linear Programming, Washington, D.C., 1955, pp. 509-565. National Bureau of Standards, Washington, D.C., 1955.

This paper is concerned with linear programming problems of the following special form:

To minimize the linear form $\sum_{ij} C_{ij} X_{ij}$ subject to the constraints

$$(1) \sum_{ij} X_{ij} \leq M_i, (2) \sum_{ij} a_{ij} X_{ij} = T_j, (3) X_{ij} \geq 0,$$

where the M_i , T_j , a_{ij} , and C_{ij} are given and the X_{ij} are to be determined. Problems of this sort arise in the analysis of machine-tool substitution possibilities, in which M_i is the total time available on the i th machine, T_j is the amount of a given "task" which must be performed, a_{ij} is the rate at which the i th machine performs the j th task and C_{ij} is the cost of performing the j th task on the i th machine. It is desired to determine the time during which the i th machine shall be performing the j th task in order to minimize the total cost.

The author derives a number of special properties of solutions of this type of problem and proposes some computational techniques for their solution which take advantage of the particular form of the constraints. The specific results are too complicated to be described in detail here. Some of them suggest similar results in transportation problems of which the above problem is a generalization. *D. Gale* (Providence, R.I.).

Rutishauser, Heinz. *Bestimmung der Eigenwerte und Eigenvektoren einer Matrix mit Hilfe des Quotienten-Differenzen-Algorithmus*. Z. Angew. Math. Phys. 6 (1955), 387-401.

This is the author's third major paper on his QD-algorithm [for the others see same Z. 5 (1954), 233-251, 496-508; MR 16, 176, 863]. Here the methods are extended to yield as a limit the eigenvector system of any

finite matrix A . Four algorithms are summarized in the form of flow charts for a computer; there is one numerical example. A principal theorem is the following: Suppose the eigenvalues of A are distinct in absolute value. Let $x_1^{(0)}, y_1^{(0)}$ be almost arbitrary vectors, and let $x_1^{(2\mu)} = A^{2\mu} x_1^{(0)}, y_1^{(2\mu)} = (A^*)^{2\mu} y_1^{(0)}$. Generate the biorthogonal vector system $x_1^{(2\mu)}, \dots, x_n^{(2\mu)}, y_1^{(2\mu)}, \dots, y_n^{(2\mu)}$ by the biorthogonalization (BO-) algorithm [C. Lanczos, J. Res. Nat. Bur. Standards **45** (1950), 255-282; MR **13**, 163; Rutishauser, Z. Angew. Math. Phys. **4** (1953), 35-36; MR **14**, 1055]. Then as $\mu \rightarrow \infty$ the vectors $x_\sigma^{(2\mu)}, y_\sigma^{(2\mu)}$ ($\sigma=1, \dots, n$) converge in direction to the eigenvectors of A and A^* , respectively.

A note added in proof gives a recommended procedure for solving the eigenvalue problem for $A: I$. Use the BO-algorithm to replace A by a similar matrix B in Jacobi (triple diagonal) form. II. Use the progressive form of the QD-algorithm to approximate all the eigenvalues of B . III. Use the algorithm C of the present paper to approximate all the eigenvectors of A . The paper also considers the eigenvalue problem for infinite symmetric matrices.

G. E. Forsythe (Los Angeles, Calif.).

★ Stiefel, Eduard L. Kernel polynomials in linear algebra and their numerical applications. Four lectures on solving linear equations and determining eigenvalues. National Bureau of Standards, Washington, D. C., 1955. 52 pp.

The author unifies a number of known iterative methods for solving linear problems with a matrix which has only real eigenvalues. The unifying principle is the theory of orthogonal polynomials over an arbitrary mass distribution on a real interval. [A related paper by Stiefel is in Comment. Math. Helv. **29** (1955), 157-179; MR **17**, 88.]

In the first section the author reviews the theory of orthogonal polynomials $\{P_i(\lambda)\}$ over a mass distribution $\rho(\lambda)d\lambda$. He shows how one can generate them by orthogonalizing the sequence $\{\lambda P_{i-1}(\lambda)\}$, a far more efficient process than orthogonalizing the powers $\{\lambda^i\}$. Although the author doesn't mention it here, this idea can be worked into a polynomial curve-fitting routine much more effective than the usual ones based on solving normal equations built from $\{\lambda^i\}$.

To solve a linear system $Ax=k$, the author considers simple iterative routines of the form $x_{t+1} = x_t + \Delta x_t$, where $\Delta x_t = (1/q_t)r_t$, $r_t = k - Ax_t$. Then $r_n = R_n(A)k$, where $R_n(\lambda) = \prod_{k=0}^{n-1} (1 - \lambda/q_k)$ is a polynomial with $R_n(0)=1$. The author poses the following problem of choosing a best strategy: Suppose the eigenvalues of A are known to lie in an interval $[a, b]$. Then it is desired to find that polynomial $R_n(\lambda)$ of maximal degree n such that $R_n(0)=1$ and such that $\int_a^b R_n(\lambda)^2 \rho(\lambda) d\lambda$ is minimized. Since one doesn't know the eigenvalues λ_i , such a requirement is about all that one can make about the smallness of $R_n(\lambda_i)$. The author solves this minimization problem exactly in terms of orthogonal polynomials, and shows various ways in which it may be used to construct algorithms which yield small r_n . He points out, however, that the intermediate residuals r_t may be quite large; i.e., these processes may be unstable. [See D. Young, J. Math. Phys. **32** (1954), 243-255; MR **15**, 650.]

The author now shows how to modify the above algorithm so that each intermediate r_t also corresponds to a best strategy, thus restoring stability. The change is to put $\Delta x_t = (1/q_t)(r_t + p_t \Delta x_{t-1})$, where p_t, q_t are scalars involved in the three-term recurrence relation for

$\{P_i(\lambda)\}$. He shows that various known algorithms correspond to special choices of $\rho(\lambda)$ — in particular, the conjugate gradient method of Lanczos, Hestenes, and Stiefel [Hestenes and Stiefel, J. Res. Nat. Bur. Standards **49** (1952), 409-436, see this paper for earlier references; MR **15**, 651.]

In another section the same ideas are applied to the calculation of eigenvalues of a matrix A with real eigenvalues. It is shown how to bring out any intermediate eigenvalue by iterations with an appropriate kernel polynomial. As a special case the author shows us the new spectroscopic method of Lanczos [J. Washington Acad. Sci. **45** (1955), 315-323; MR **17**, 669]. The final section applies the quotient-difference algorithm of Rutishauser [the paper reviewed above; see this for earlier references] to generating the orthogonal polynomials.

This valuable material should be available to every mathematician concerned with numerical methods in linear algebra. [However, the availability of this multi-lithed report may be limited.] Its reader will see — probably for the first time — the unity behind the gradient methods, Richardson's method, and the various conjugate gradient methods. And he will learn how to improve his technique of polynomial curve fitting. One wonders just how far Stiefel's exposition and methods could be carried through for matrices with complex eigenvalues. After all, a limitation to real eigenvalues is practically a limitation to symmetric matrices.

The reviewer's copy contains a slip-sheet with two corrections: For $n=0$ replace formulas (47), (73) by $\Delta x_0 = 2(b+a)^{-1}r_0$; $\Delta r_0 = -2(b+a)^{-1}Ar_0$, respectively.

This review has been adapted with permission from the review in Math. Tables Aids Comput. **9** (1955), 199-200. G. E. Forsythe (Los Angeles, Calif.).

Crockett, Jean Bronfenbrenner; and Chernoff, Herman. Gradient methods of maximization. Pacific J. Math. **5** (1955), 33-50.

In the present paper the authors discuss various gradient methods based upon an arbitrary metric. Given a positive definite matrix B , the gradient of f is given by $B^{-1}\nabla f$, where ∇f is the gradient relative to a Euclidean metric. The matrix B may be a function of x . By taking B as the matrix of second derivatives of f , one obtains Newton's method. The authors give a discussion of the ratio of convergence, together with some remarks concerning their computational experience in using these methods. Some discussion is made with regard to the effects of overcorrecting and undercorrecting in the use of gradient methods. M. R. Hestenes.

Householder, Alston S. On the convergence of matrix iterations. Oak Ridge National Laboratory, Oak Ridge, Tenn., Rep. ORNL 1883 (1955), 47 pp.

This is a unified exposition with proofs of results by many authors on matrix norms and eigenvalues, and on the convergence of linear iterations in n -dimensional space. There are a number of new theorems, especially in connection with the 'g-norms' introduced earlier [same Rep. ORNL 1756 (1954); MR **16**, 211]. Single-step (S) and total-step (T) iterations are compared. In particular, when a coefficient matrix has Young's property (A), a simple matrix proof is given that (T) converges twice as fast as (S) for a certain ordering [D. Young, Trans. Amer. Math. Soc. **76** (1954), 92-111; MR **15**, 562].

Let $A = A^T = S - B_1 - B_2$, where $S = S^T$. The following new theorem is typical: Suppose $S + B_2 - B_1^T$ is positive

definite. Then the generalized (S) iteration defined by

$$(S-B_1)x_{p+1}-B_2x_p=h$$

converges if and only if A is positive definite. There are 29 references. *G. E. Forsythe* (Los Angeles, Calif.).

Loo, Win; et Kwan, Chao-Chih. *La méthode de col dans le problème de relaxation.* Acta Math. Sinica 5 (1955), 497-504. (Chinese. French summary)

Let A be a positive definite symmetric operator in a real Hilbert space H such that $m(x, x) \leq (Ax, x) \leq M(x, x)$ for all $x \in H$, where $0 \leq m \leq M < +\infty$. This paper deals with an iterative process for solving the equation $Ax=b$ ($b \in H$). The sequence of successive approximations $x_0, x_1, \dots, x_n, \dots$ is constructed in the following way. Let k be a fixed positive number < 1 . After $r_n = Ax_n - b$ is obtained, choose a finite-dimensional linear subspace H_{n+1} of H such that $\|P_{n+1}r_n\|^2 \geq k\|r_n\|^2$, where P_{n+1} denotes the orthogonal projection on H_{n+1} . Choose $z_{n+1} \in H_{n+1}$ so that it maximizes $(r_n, z)^2 (Az, z)^{-1}$ over H_{n+1} . Then take $x_{n+1} = x_n - c_n z_{n+1}$ with $c_n = (r_n, z_{n+1}) \times (Az_{n+1}, z_{n+1})^{-1}$. It is proved that for any arbitrary initial x_0 ,

$$\|x_n - A^{-1}b\|^2 \leq \frac{M}{m} \left(1 - \frac{m}{M}k\right)^n \|x_0 - A^{-1}b\|^2$$

holds, whence $\lim x_n = A^{-1}b$. The new feature of this iterative scheme lies in the selection of a finite-dimensional linear subspace H_n at each step. The authors point out that the convergence in their process is slower than that in the method of steepest descent as treated by L. V. Kantorovič [Uspehi Mat. Nauk (N.S.) 3 (1948), no. 6(28), 89-185; MR 10, 380]. *Ky Fan* (Notre Dame, Ind.).

Arnold, Kurt. *Das abgekürzte Eggert'sche Verfahren zum Ausgleichen grosser geodätischer Systeme nach der Methode der kleinsten Quadrate.* Deutsche Akad. Wiss. Berlin. Veröff. Geodät. Inst. Potsdam no. 7 (1955), ii+46 pp. (13 plates).

In triangulation the direction measurements of a large net are adjusted by the method of least squares. O Eggert [Verh. Baltischen Geodät. Kommission, Helsinki 9 (1936), 27-28, 114-119; Bull. Géodésique 1936, 474-480] proposed a new adjustment process in which subnets are first adjusted separately. The present author reviews the literature and then presents a first general proof of the validity of Eggert's method. There is a discussion of the weights to be given to observations common to several subnets. There are two complete numerical examples and 20 references. *G. E. Forsythe* (Los Angeles, Calif.).

Babuška, Ivo. *Über eine numerische Lösung von vollständig regulären Systemen linearer Gleichungen und ihre Applikation auf die statische Lösung von Rahmen-tragwerken.* Časopis Pěst. Mat. 80 (1955), 60-88. (Czech. Russian and German summaries)

C. V. Klouček [Distribution of deformation (a new method of structural analysis), Prague, 1949; MR 11, 60] proposed an iterative "method of distributed deformations" for solving certain linear algebraic equation systems arising in the static solution of structural frameworks. The present author gives a well-organized mathematical exposition of the method, with proofs and several numerical examples.

The systems considered have symmetric matrices A with $a_{ij} = \alpha_i \sum_{j \in G_i} |a_{ij}|$, where all $\alpha_i > 1$. For infinite systems also $0 < k < a_{ii} < K < \infty$. The author calls such

systems "completely regular". He represents such a matrix by a graph. Each unknown x_i is represented by a point P_i , and two points P_i, P_j are joined by a line if $a_{ij} \neq 0$. A "closed" system has a simple closed curve in its graph; all others are "open". A complicated closed algorithm with quotients and differences is given for the solution of a finite, completely regular, open system $S: Ax=b$ with only one non-zero component in b . Klouček's method is then developed for any finite, completely regular, closed system S with the same type of b . The essence of the method is to replace S by an equivalent infinite, completely regular, open system S^* , and to find the solution x_n^* of the n th finite segment S_n^* of S^* by the above algorithm. As $n \rightarrow \infty$, x_n^* converges to the solution of S . *G. E. Forsythe*.

Zimm, Bruno H.; Roe, Glenn M.; and Epstein, Leo F. *Solution of a characteristic value problem from the theory of chain molecules.* J. Chem. Phys. 24 (1956), 279-280.

An eigenvalue problem encountered in the dynamical theory of chain molecules is

$$f_{-1} \alpha''(s) (|r-s|)^{-1} ds = -\lambda \alpha(r), \quad \alpha'(\pm 1) = 0.$$

This is solved by three methods: use of a Fourier series for α , expansion of α in associated Legendre polynomials P_m^2 , and by a variation method. The eight smallest eigenvalues are calculated explicitly and an approximate formula is found for the remaining ones. Formulas are found also for the eigenfunctions. *Author's summary*.

Morgenstern, Dietrich. *Statistische Begründung numerischer Quadratur.* Math. Nachr. 13 (1955), 161-164.

Consider an ensemble of real-valued functions $x(t)$ for $0 \leq t \leq a$, with Wiener measure in the ensemble. To select abscissas t_i and weights w_i for a linear quadrature formula, the author proposes minimizing the expected value $E(D^2)$ of D^2 , where

$$D = F - \sum_{i=1}^n w_i x(t_i), \quad F = \int_0^a x(t) dt.$$

Given that $x(t) = 0$, he notes that

$$E[x(t)^2] = t, \quad E[x(t)|x(s)] = ts^{-1}x(s) \quad (t \leq s), \\ E(F) = 0, \quad E(F^2) = 3^{-1}a^3.$$

Using these relations, he shows that $E(D^2)$ is minimized when $t_1 = (2n)^{-1}$, $t_{i+1} - t_i = n^{-1}$, $t_n = 1 - (2n)^{-1}$, and all w_i are equal. This "MacLaurin" midpoint-quadrature formula is not exact for a parabola.

Reviewer's note: For random functions with a more general covariance function, minimizing $E(D^2)$ has been considered also by Blanc [C. R. Acad. Sci. Paris 233 (1951), 726-727; MR 13, 368] and Blanc and Liniger [Z. Angew. Math. Mech. 35 (1955), 121-130; MR 16, 1154]. *G. E. Forsythe* (Los Angeles, Calif.).

Young, Robert L. *Report on experiments in approximating the solution of a differential equation.* J. Assoc. Comput. Mach. 3 (1956), 26-28.

Step-by-step solution of Mathieu's equation, for one combination of parameters, was carried out using a fourth-order Runge-Kutta process, and the accuracy compared with the results of solving an associated Volterra-type integral equation using Simpson's rule. The first method had larger errors at interval h than the second at interval $2h$, and the cause is thought to be the accumulation of rounding errors, rather than the presence of truncation error. *L. Fox* (Teddington).

Lotkin, M.; and Browne, H. N. On the accuracy of the adjoint method of differential corrections. *Amer. Math. Monthly* 63 (1956), 97-105.

The "method of adjoints" is widely used in ballistics and related fields to compute the differential effects of small perturbations upon the solutions of the ordinary differential equations there arising. The method invokes the use of partial derivatives, often of empirical functions, and usually involves numerical integration. The authors examine under reasonably general conditions, the maximum numerical errors that can arise in the perturbed solutions, in terms of errors in these partial derivatives, and of the size of integration step, and number of such steps. Specially simple explicit results are obtained for the case where the system of ordinary differential equations reduces to a single equation.

A. A. Bennett.

Karmišin, A. V. A method of solution of a system of three-term algebraic equations and its applications to the solution of some problems of mathematical physics. *Vestnik Moskov. Univ.* 10 (1955), no. 8, 39-45. (Russian)

Consider the system of three-term difference equations $y_t + \alpha_t y_{t+1} + \beta_{t-1} y_{t-1} = \gamma_t$ with unknowns y_t . By Cramer's rule one can express each y_t as a quotient of determinants $\Delta^{(t)}/\Delta_1$. The author sets up recurrence relations by which the determinants $\Delta^{(t)}$, Δ_1 can be evaluated, presumably by desk calculation. He shows how to evaluate the α_t , β_t , and γ_t for a difference approach to the differential equation $y'' + p(x)y' + q(x)y = f(x)$. For a boundary-value problem for an elliptic partial differential equation in a rectangle the author interprets the associated difference equations in $(n-1)(m-1)$ unknowns as $(m-1)$ systems of three-term difference equations in $(n-1)$ unknowns each. It is not stated how this helps to solve the equations numerically. There is a numerical example with 10 unknowns y_t .

G. E. Forsythe (Los Angeles, Calif.).

Daymond, S. D. The principal frequencies of vibrating systems with elliptic boundaries. *Quart. J. Mech. Appl. Math.* 8 (1955), 361-372.

Let R be a plane region of area π bounded by an ellipse c of eccentricity e . The author considers the two eigenvalue problems

$$(1) \quad \nabla^2 \phi + \lambda \phi = 0, \quad \phi = 0 \text{ on } c,$$

$$(2) \quad \nabla^2 \phi + \mu \phi = 0, \quad \frac{\partial \phi}{\partial n} = 0 \text{ on } c \quad (\mu \neq 0),$$

concentrating on the smallest λ , μ from the positive sequence of eigenvalues possible in each case. Solution involves considering the zeros of certain Mathieu functions and their derivatives. λ and μ are tabulated to 4 or more decimals for $e=0(0.1)1$ and $(1-e^2)^{1/2}=0(0.1)1$. As e increases from 0 to 1, λ increases to ∞ and μ decreases to 0, both monotonically. However, as $e \rightarrow 1$, $\lambda(1-e^2)^{1/2} \rightarrow \frac{1}{2}\pi^2 = 2.46740$ from above, and $\mu(1-e^2)^{-1/2} \rightarrow 3.55928$ from below, both monotonically (on page 362, line 31, for $m_1=0$ read $m_1=3.55928$). λ and μ (especially λ) change rather slowly as e goes from 0 to 0.9. For an elliptical region R with area A the corresponding $\bar{\lambda}$ and $\bar{\mu}$ are given, for each eccentricity e , by

$$\bar{\lambda} = \frac{\pi}{A} \lambda, \quad \bar{\mu} = \frac{\pi}{A} \mu.$$

M. A. Hyman (Philadelphia, Pa.).

Litvinov, N. V. On the solution of an infinite system of finite-difference equations of the theory of elasticity for a strip. *Ukrain. Mat. Ž.* 7 (1955), 188-206 (1 plate). (Russian)

A basic problem in the theory of elasticity is formulated here for an infinite strip. The solution is reduced to determination of the stress function F , satisfying the bi-harmonic equation $\nabla^2 \nabla^2 F = 0$, with the boundary conditions prescribing the values of F and $\partial F / \partial n$. For numerical calculation the partial differential equation is replaced by a system of difference equations for a square mesh of points. In the matrix form $AF + V = 0$, where A is the matrix of coefficients, F is a column of unknown values, and V is a column of constants. The formal solution is $F = -A^{-1}V$. The elements of A are, themselves, infinite matrices (cells). An element of the latter, therefore, carries four indices; i.e., $A_{jk}{}^{i'k'}$; $i, i' = 1, \dots, n'$; $j, k = -\infty, \dots, +\infty$. Here i', k refer to a mesh point entering the equation centered at the point i, j . The inversion of A by the usual methods is extremely laborious. The author claims that the method developed in the paper is considerably shorter. It is based on a direct process of successive elimination of "cells", analogous to the Gaussian algorithm commonly applied to finite matrices. This operation is carried out in the space i, i' . Since the rows of the infinite cells are identical sequences of numbers, all the numerical operations involving the cells can be replaced by operations with arbitrarily chosen columns and rows. The intermediate process of inverting the infinite cells is accomplished by a special matrix transformation in the space j, k . A detailed numerical example is included.

B. Garfinkel (Aberdeen, Md.).

Vasil'ev, V. V. On a class of nonlinear integral equations. *Irkutsk. Gos. Univ. Trudy.* 8 (1953), no. 1, 22-27. (Russian)

To obtain an approximate solution of

$$F(x, \varphi(x)) = \varphi(x) + \int_a^b K(x, s) f(s, \varphi(s)) ds$$

with $f(s, u)$ a polynomial in u and

$$K(x, s) = \sum_{i=1}^n \varphi_i(x) \varphi_i(s),$$

the author replaces the unknown function $\varphi(s)$ by the two numbers $y = \varphi(s_1)$, $z = \varphi(s_2)$, uses a 2-point quadrature formula and solves the resulting algebraic system graphically.

M. Golomb (Lafayette, Ind.).

Kohler, K. Die allgemeine Doppelzett-Fluchtentafel. *Z. Angew. Math. Mech.* 35 (1955), 476-478.

The author develops the general double-alignment nomogram with a straight reference line in a determinantal form referred to two separate parallel-line coordinate systems with a common base line. He conceives his generalization, in which for four variables the function-scales may be curved lines, as an extension of the double-Z (or double-N) form in which all function-scales are straight lines; an immediate further generalization is the replacement of the curved function-scales in one variable by a grid in two variables. The results are well-known; they are more readily deduced and are then more tractable in the dual cartesian coordinate system [cf., e.g., Allcock, Jones, and Michel, *The nomogram*, 4th ed., Pitman, New York, 1950, Ch. II; MR 12, 362].

J. G. L. Michel.

Horváth, F. Technique of nomography. *Magyar Tud. Akad. Alkalm. Mat. Int. Közl.* 3 (1954), 343-352 (1955). (Hungarian. Russian and English summaries)

The author reviews the results achieved in the technique of nomography by the department for numerical and

graphical methods of the Institute for Applied Mathematics of the Hungarian Academy of Sciences.

From the English summary.

See also: Parodi, p. 703; Stesin, p. 749; Fichera, p. 770; Nakata and Fujita, p. 804.

Tables

Paszkowski, S. New methods of tabulating functions. *Zastos. Mat.* 2 (1955), 232-262 (2 inserts). (Polish. Russian and English summaries)

The purpose of the paper is to illustrate the author's methods of condensing tables of functions without loss of information, at the same time permitting the user only pencil and paper methods. Sample tables are given and methods illustrated. There is even a specimen multiplication table. The general method is to use an interpolation of the form

$$x(a+hk) = x(a) \pm 10A^{(a) + \log h + v(a, h)},$$

where $x(a)$ is a tabulated value of the function x , h is the interval of tabulation and $A(a)$ is a tabulated auxiliary function. The function $v(a, h)$ is computed by a nomogram. *D. H. Lehmer* (Berkeley, Calif.).

See also: Mitropol'skii, p. 702; Sokolov, p. 731; Dingle, p. 737; Makabe and Morimura, p. 756; Chandra sekhar, Agarwala and Chakrabarty, p. 758; Daymond, p. 792.

Mathematical Machines

*Bückner, Hans. *Moderne Rechenmaschinen*. Handbuch der Physik. Bd. II., pp. 471-498. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1955. DM 88.00.

This review article is divided into two sections, covering analogue and digital computers respectively. The emphasis is, rightly, on electronic internally-programmed digital calculators. The history of the subject is briefly outlined, from Babbage and Lady Lovelace till about 1954. A discussion of programming with examples of coding is based on the system used for the "EDSAC" in Cambridge, England. The logical structure of arithmetic operations in computers, use of binary numbers, representation of truth functions are amongst the theoretical

topics. A section is devoted to the various kinds of memory employed. *W. Freiburger* (Providence, R.I.).

*Eckert, W. J.; and Jones, Rebecca. *Faster, faster*. A simple description of a giant electronic calculator and the problems it solves. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1955. vii+160 pp. (1 plate). \$3.75.

Gurov, V. V.; Kogan, B. Ya.; Talancev, A. D.; and Trapeznikov, V. A. The new electronic analogue apparatus of the Institute of Automatics and Telemechanics of the Academy of Sciences of the USSR. *Avtomat. i Telemekh.* 17 (1956), 19-35. (Russian)

Lemann, I. On design and construction of small automatic computing machines with programmed control in the Technische Hochschule in Dresden. *Avtomat. i Telemekh.* 17 (1956), 3-18. (Russian)

Frenkiel, François N. Possibilities and significance of high-speed computing in meteorology. *J. Washington Acad. Sci.* 46 (1956), 33-37.

Kogan, B. Ya. The methodology of set-up and solution of problems on electronic analogue computers. *Avtomat. i Telemekh.* 17 (1956), 36-52. (Russian)

Hoffmann, Hans. *Aufbau und Wirkungsweise neuzeitlicher Integrieranlagen*. *Elektrotech. Z.* 77 (1956), 77-83.

Levšunov, M. T. On the use of the arithmometer for extraction of square roots. *Stavropol. Gos. Ped. Inst. Sb. Nauč. Trud.* 1952, no. 8, 135-143 (1953). (Russian)

Medgyessy, P. Product integration, Fourier-synthesis and similar operations carried out by means of a square planimeter and a new apparatus. *Magyar Tud. Akad. Alkalm. Mat. Int. Közl.* 3 (1954), 129-137 (1955). (Hungarian. Russian and English summaries)

Van Brocklin, G. R., Jr.; and Murray, P. G. A polar-planimeter method for determining the probability of hitting a target. *Operations Res.* 4 (1956), 87-91.

See also: Schützenberger, p. 702; Nicholson and Jeanel, p. 789.

RELATIVITY

Winogradzki, Judith. Sur les "identités de Bianchi" de la théorie unitaire d'Einstein-Schrödinger. *C. R. Acad. Sci. Paris* 242 (1956), 74-76.

Put
(1) $*\Gamma_{\lambda\mu} = \Gamma_{\lambda\mu} + \delta_{\lambda}^{\nu} u_{\nu}$
and denote by \mathfrak{U} any density which is transformed by (1) according to

(2) $*\mathfrak{U} = \mathfrak{U} - 2g^{(\lambda\nu)} \partial_{\nu} u_{\lambda}$,
* $g^{\lambda\nu}$ being the inverse to $g_{\lambda\mu}$. If \mathfrak{U}' is another such density, then for $\mathfrak{B} = (\mathfrak{U} - \mathfrak{U}')$

$$\int_D \mathfrak{B} d\tau$$

is independent of (2) and therefore, according to Noether

[Nachr. Ges. Wiss. Göttingen. Math.-Phys. Kl. 1918, 235-257] we must have four identities of the first order and four algebraic identities of the form

$$\delta_{\lambda}^{\nu} \frac{\partial \mathfrak{B}}{\Gamma_{\lambda\mu}} = 0$$

or

$$(3) \quad \delta_{\lambda}^{\nu} \frac{\partial \mathfrak{U}}{\partial \Gamma_{\lambda\mu}} = \delta_{\lambda}^{\nu} \frac{\partial \mathfrak{U}'}{\partial \Gamma_{\lambda\mu}}$$

If we put in (3) $\mathfrak{U}' = \frac{1}{2} g^{(\lambda\nu)} \partial_{\nu} \Gamma_{\lambda}$, according to the author

$$(4) \quad \delta_{\lambda}^{\nu} \frac{\partial \mathfrak{U}}{\partial \Gamma_{\lambda\mu}} - 2\partial_{\lambda} g^{(\mu\nu)} = 0$$

for any density \mathfrak{U} satisfying (2).

V. Hlavatý.

Martuscelli, Laura. Sopra una possibile modificazione della teoria unitaria di Einstein. *Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat.* (3) 19(88) (1955), 607-615.

[In this review we denote by C and M the following papers of Hlavatý, J. *Rational Mech. Anal.* 3 (1954), 103-146; *Nieuw Arch. Wiskunde* (3) 2 (1954), 103-114; the symbolism used in this review is taken from these papers; MR 15, 654; 16, 408.] The author looks for quadratic skew-symmetric tensors in the unified field theory which satisfy the first set of Maxwell equations and might be, according to the author, identified with the electromagnetic field tensor.

1) Starting with $F^{ab} = \frac{1}{2}|g|^{-1/2}\epsilon^{abcd}g_{cd}$, one finds easily (C) that by virtue of $S_a = 0$ the tensor $m_{ab} = \frac{1}{2}R_{abcd}g^{cd}$ satisfies the first set of Maxwell equations

$$(1) \quad \partial_{[a}m_{b]} = 0.$$

The author defines the current vector γ^a by

$$(2) \quad \gamma^a \stackrel{\text{def}}{=} \tilde{D}_b F^{ab},$$

where \tilde{D} denotes covariant derivative with respect to Γ_{ab}^c . (The definition (2) does not reduce to the usual one in the case of Minkowskian g_{ab}).

2) According to the Einstein conditions

$$(3) \quad \partial_{[a}R_{b]} = 0,$$

$R_{[ab]}$ could be identified with the electromagnetic field tensor. The current vector is then defined by $\gamma^a = \tilde{D}_b R^{ab}$. In the first approximation we have $\gamma^a = 0$.

3) From C, (6.4)b and (3) one obtains $\partial_{[a}D_{b]}S_{cd} = 0$.

According to the author $\partial_{[a}D_{b]}S_{cd} = 0$, so that $D_a S_{ab}$ could also be identified with the electromagnetic field.

V. Hlavatý (Bloomington, Ind.).

Pham Tan Hoang. Sur les équations approchées de la théorie unitaire d'Einstein-Schrödinger. *C. R. Acad. Sci. Paris* 242 (1956), 738-740.

Let g^{ab} be the inverse tensor to g_{ab} . Put

$$g_{ab} = \gamma_{ab} + \varphi_{ab}, \quad g^{ab} = h^{ab} + f^{ab},$$

(γ and h are symmetric, φ and f skew symmetric). The author takes for the fundamental tensor the sum of

$$b^{\mu\nu} = (h/g)^{1/2}h^{\mu\nu}, \quad q^{\mu\nu} = (h/g)^{1/2}f^{\mu\nu},$$

and finds in terms of this tensor the approximation of the Einstein-Schrödinger equations and the corresponding momentum-energy tensor. [See also Hlavatý, *Proc. Nat. Acad. Sci. U.S.A.* 39 (1953), 501-506; MR 14, 1132.]

V. Hlavatý (Bloomington, Ind.).

Pham Tan Hoang. Sur le choix de la métrique en théorie unitaire. *C. R. Acad. Sci. Paris* 241 (1955), 1919-1921.

The author discusses the determination of an invariant that enters into the choice of the metric in the Einstein-Schrödinger unified field theory.

M. Wyman.

Krempaský, Július. The strain tensor in space and time as a result of motion. *Mat.-Fyz. Časopis. Slovensk. Akad. Vied* 5 (1955), 124-131. (Slovak. Russian summary)

This paper deals with a tensor that characterizes the known deformations of space (contractions) and time (dilations) in space-time, these deformations being generated only through motion. The author makes use

only of cartesian, respectively pseudocartesian, tensors and obtains all his results by quite elementary means. He shows the full analogy of this tensor to the strain tensor that characterizes the homogeneous deformation of deformable bodies in euclidean space. Finally he deduces all kinematical relations of special relativity from this tensor. The whole statement is expressed in the symbolic notation.

T. P. Andelić (Belgrade).

Majorana, Quirino. Sulla cinematica relativistica. *Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat.* (8) 18 (1955), 462-466.

Kraichnan, Robert H. Possibility of unequal gravitational and inertial masses. *Phys. Rev.* (2) 101 (1956), 482-488.

Lorentz invariant theories are considered in which a tensor gravitational field is coupled to its own stress tensor as well as to a stress tensor constructed out of extra field variables representing matter. On a previous paper [Phys. Rev. (2) 98 (1955), 1118-1122; MR 16, 1166] the author has shown that the resulting feedback principle which is operative for the gravitational field by itself implies necessarily that the field equations (assumed derivable from a Lagrangian principle) are generally covariant in regions where matter is absent. The present paper goes on to show that unless the matter Lagrangian is also generally covariant (i.e., unless the coupling constant is the same for both the matter stress and the gravitational stress) the solutions of the resulting field equations are so restricted by extra constraints as to be either trivial or completely non-physical.

B. S. De Witt (Chapel Hill, N.C.).

Rayner, C. B. Whitehead's law of gravitation in a space-time of constant curvature. *Proc. Phys. Soc. Sect. B.* 68 (1955), 944-950.

Whitehead's law of gravitation for a discrete set of particles is generalized, and the line-element is found for the gravitational field due to a finite number of particles of arbitrary proper mass and having arbitrary world-lines in a space-time of constant curvature. In this calculation the author corrects a formula for the generalized potential function given previously by G. Temple.

The line-element is then extended to describe the gravitational field of a distribution of matter with proper density and velocity as continuous functions of position and time in space-time of constant curvature. Finally the author discusses certain cosmological aspects of his results.

A. G. Walker (Liverpool).

Castoldi, Luigi. Attorno a una teoria sulla interazione tra masse in movimento. *Boll. Un. Mat. Ital.* (3) 10 (1955), 328-331.

Referring to a recent paper by S. Melone on gravitational field theory [same Boll. (3) 10 (1955), 68-74; MR 17, 95], the author points out that the equation of continuity $\Delta_t I^t = 0$, where $I^t = \rho v^t$, ρ being the density and v^t being the four-dimensional velocity of matter, can be written as an equation of Maxwellian form $I^t = \Delta_j F^{jt}$, where the skew-symmetric tensor F^{jt} is determined from I^t except for an arbitrary additive curl. Melone's theory consists in adding as gravitational field equations the remaining equations of Maxwellian form $\epsilon^{ijkl}\Delta_j F_{kl} = 0$, thus completing the analogy with the electromagnetic field.

A. J. McConnell (Dublin).

Fantappiè, Luigi. Su una nuova teoria di "relatività finale". Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 17 (1954), 158-165.

The Galilean group G_{3+1}^{10} (the 10-parameter group of motions in space and in time), which is the basic group associated with classical physics, is the limiting case as the velocity of light c tends to infinity of the Lorentz group L_{3+1}^{10} (the 10-parameter group of motions in space-time), which is the basic group associated with special relativity theory. The author shows that the Lorentz group is itself the limiting case (as a constant

R tends to infinity) of a new group Fn_{3+1}^{10} , which is the 10-parameter group of motions in a de Sitter universe of radius R . Moreover, this last group cannot be regarded in the same way as the limiting case of a further new group, so that Fn_{3+1}^{10} may be taken to be the basic group associated with a theory of "final relativity", which reduces to special relativity theory when $R \rightarrow \infty$. From the structure of this final group the author proposes to determine the fundamental operators and the fundamental quantum magnitudes of our physical universe in their most general form. *A. J. McConnell* (Dublin).

MECHANICS

★**Rubinowicz, Wojciech; i Królikowski, Wojciech.** *Mechanika teoretyczna.* [Theoretical mechanics.] Państwowe Wydawnictwo Naukowe, Warszawa, 1955. 422 pp. (5 plates). 26.00 zł.

Eremeev, N. V. Plane hinged four-bar linkages with dynamically defined motion. Moskov. Gos. Univ. Uč. Zap. 172 (1954). Meh. 5, 227-240. (Russian)

Kislicyn, S. G. On the theoretical form of the profile of a tooth cut by an evolute cutter. Leningrad. Gos. Ped. Inst. Uč. Zap. 89 (1953), 145-151. (Russian)

Selig, F. Allgemeine Sätze über Momentankräfte. Z. Angew. Math. Mech. 35 (1955), 464-465.

Kurze Notiz über die Anwendung der von Miseschen Motorrechnung auf Stoßprobleme. Beweis der Carnotschen Sätze. *O. Bottema* (Delft).

Četaev, N. G. On a property of the Poincaré equations. Prikl. Mat. Meh. 19 (1955), 513-515. (Russian)

Starting from the Poincaré equations of motion for a holonomic dynamical system [Poincaré, C.R. Acad. Sci. Paris 132 (1901), 369-371; N. G. Četaev: Dokl. Akad. Nauk SSSR 1928, 103-104; Prikl. Mat. Meh. 5 (1941), 253-262; MR 4, 225] and from variational equations corresponding to them, the author derives a rule for the stability of motion. This rule states that the motion should be stable when for the limited generalized impulses the variational equations of the problem possess a sign-definite integral.

As an example he treats the problem of the stability of helicoidal motions of rigid bodies in a fluid for the case when the mass distribution of the body, as well as its bounding surface, has three mutually orthogonal planes of symmetry. *T. P. Andelić* (Belgrade).

Clauser, Emilio. Equazioni dinamiche rappresentate da autoparallele di spazi non riemanniani. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 18 (1955), 495-501.

A holonomic dynamical system with n degrees of freedom q^i whose kinetic energy is expressible as a quadratic form in the velocity components \dot{q}^i defines a Riemannian metric in the configuration space of the q^i . The equations of Lagrange are compared with the differential equations of the "paths" of this space, the paths being defined by means of connection coefficients Γ_{ik}^j which differ from the Christoffel symbols of the Riemannian metric by a tensor γ_{ik}^j . The latter must satisfy certain conditions in order that the trajectories (as given by the equations of Lagrange) coincide with the paths. It may happen that the γ_{ik}^j depend on the q^i , in

which case the paths are more general than those of non-Riemannian geometry. [Reviewer's remark: Such spaces have in fact been investigated: cf. J. Douglas, Ann. of Math. (2) 29 (1928), 143-168.] A connection similar to that of Weyl is introduced: if the coefficients of the latter depend only on the q^i , the Weyl space turns out to be conformal to the original Riemannian space. The system is then conservative, which is in agreement with the classical principle of least action. *H. Rund* (Durban).

Carini, Giovanni. Sull'equazione dell'energia nella dinamica del punto a massa variabile. Boll. Un. Mat. Ital. (3) 10 (1955), 224-228.

Si stabilisce l'equazione dell'energia della dinamica del punto a massa variabile. Quindi, dalla combinazione dell'equazione trovata col principio di relatività, si deduce l'equazione newtoniana nella forma in cui è stata posta da Sommerfeld. *Author's summary.*

Bautin, N. N. Dynamical models of nonfree watch movements. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk 1955, no. 10, 60-83. (Russian)

Barthel, Josef. Die Bewegungen des beiderseits eingespannten idealen Seiles. Ann. Univ. Sarav. 3 (1954), 352-359 (1955).

The equations of motion are combined with some elementary differential-geometric conditions to reduce the equations determining the motion of an inextensible, flexible cable suspended between two points to convenient form, particularly for the case of constant or quasi-constant force field. For small vibrations approximations are made which simplify the equations further and split them into two equations describing motion in the rest plane and one for the perpendicular (horizontal) motion. For a gravitational field the vertical and horizontal small stationary vibrations are found. [The "-" sign in equation (6b) should be "+".] *H. D. Block*

Egerváry, J. Über eine Anwendung der Hunyadi-Scholtzschschen Matrizen in der Theorie der ebenen und räumlichen Fachwerken. Magyar Tud. Akad. Alkalm. Mat. Int. Közl. 3 (1954), 289-300 (1955). (Hungarian. Russian and German summaries)

Egerváry, E. On the application of the matrix theory to the calculation of chain bridges. Magyar Tud. Akad. Alkalm. Mat. Int. Közl. 3 (1954), 9-23 (1955). (Hungarian. Russian and English summaries)
Hungarian version of a paper in Acta Tech. Acad. Sci. Hungar. 11 (1955), 241-256; MR 16, 894.

Tasnády, I. Bemerkungen zur Abhandlung von J. Eger-vary über die Anwendung des Matrizenkalküls bei der Konstruktion von Kettenbrücken. Magyar Tud. Akad. Alkalm. Mat. Int. Közl. 3 (1954), 25-30 (1955). (Hungarian. Russian and German summaries)
Remarks on the paper reviewed above.

See also: Eger-vary, p. 703.

Fluid Mechanics, Acoustics

★ Milne-Thomson, L. M. Theoretical hydrodynamics. 3rd ed. The Macmillan Company, New York, 1956. xxiii+632 pp. (4 plates). \$9.80.

"Aside from rearrangements and new methods of presentation this third edition differs from the second [1950; MR 11, 471] by three important additions: C. G. Darwin's interpretation of virtual mass [Proc. Cambridge Philos. Soc. 49 (1953), 342-354; MR 14, 1027]; M. Schiffman's method of reflection across free streamlines [Comm. Pure Appl. Math. 1 (1948), 89-99; 2 (1949), 1-11; MR 11, 220]; F. John's treatment of potential flow with a free surface [ibid. 6 (1953), 497-503; MR 15, 570]."

Ikenberry, E.; and Truesdell, C. On the pressures and the flux of energy in a gas according to Maxwell's kinetic theory. I. J. Rational Mech. Anal. 5 (1956), 1-54.

Truesdell, C. On the pressures and the flux of energy in a gas according to Maxwell's kinetic theory. II. J. Rational Mech. Anal. 5 (1956), 55-128.

Unter der bescheidenen Überschrift dieser zwei Teile (erster Teil=Kap. I, II; zweiter Teil=Kap. III-VII) verbirgt sich eine monumentale Arbeit, die die Maxwell-Boltzmann'sche Gleichung (M.B.G.) konsequent (ohne heuristisch-physikalische Betrachtungen wie sonst oft) als Grundlage verwendet. Im allgemeinen liegt der Fall Maxwell'scher Moleküle zugrunde. Von den umfangreichen Untersuchungen kann nur das wichtigste referiert werden:

Im ersten Kapitel werden die der M.B.G. äquivalenten Gleichungen für die Momente (in Bezug auf den Geschwindigkeitsraum) verwendet, um durch ein Iterationsverfahren gelöst zu werden, dessen erster und zweiter Schritt die Navier-Stokes'schen Gleichungen bzw. die Burnett'schen Gleichungen ergeben, und welches zur Ehre Maxwell's, der diese Behandlungsweise vorgeahnt haben könnte, als Maxwell'sche Iteration bezeichnet wird. Es geht von lokal-Maxwell'sche Verteilungen aus und verwendet die Momente in wachsender Ordnungszahl und wird durch besondere „sphärische Momente“ einfacher gestaltet. An Anfangsdaten werden nur die klassischen physikalischen Größen (Dichte, Geschwindigkeit, Temperatur und Druck) verwendet.

Kapitel zwei behandelt Abarten dieses Iterationsverfahrens oder der Anfangsiteration.

Im dritten Kapitel werden allgemein interessierende Ausführungen über derartige Typen von Iterationsverfahren, die der Verfasser als „Differential-Iteration“ bezeichnet, und ihr Limesverhalten bei $t \rightarrow \infty$ gemacht. An dem Modell-Gleichungssystem

$$-\frac{1}{\mu} \dot{p}_m = \frac{d\dot{p}_m}{dt} + \alpha \dot{p}_m + \beta \dot{p}_{m+1} + \delta_{0m} E \quad (m=0, 1, \dots)$$

wird das Verhalten der Lösungen durch die Differential-

iteration

$$-\frac{1}{\mu} \dot{p}_m^{(n+1)} = \frac{d\dot{p}_m^{(n)}}{dt} + \alpha \dot{p}_m^{(n)} + \beta \dot{p}_{m+1}^{(n)} + \delta_{0m} E$$

sowie das Verhalten bei $t \rightarrow \infty$ und $\mu \rightarrow +0$ studiert und die analogen Ergebnisse für die kinetische Theorie als Vermutungen ausgesprochen.

Im vierten Kapitel werden Integral-Iterationsverfahren behandelt, bei denen im Gegensatz zu Kap. II bei jedem Schritt ein System partieller Differentialgleichungen gelöst werden muß und als Anfangsdaten alle Momente eingehen. Als Beispiel wird die Scher-Strömung (shearing-flow) behandelt, deren exakte Lösung im fünften Kapitel gegeben wird. Das ist eine ausführlich diskutierte exakte Lösung der M.B.G.(11), bei der alle Momente mit Ordnung ≥ 2 räumlich konstant aber zeitlich veränderlich sind.

Im sechsten Kapitel wird unter anderen auf Grund des Differentialgleichungssystems für die Momente gezeigt, daß im räumlich homogenen Fall diese Momente gegen die Werte der Gleichgewichtsverteilung streben [woraus die „schwache Konvergenz“ der Dichte selbst folgt. Ref.], was einen wichtigen Beitrag zur Lösung dieses klassischen analytischen Problems darstellt.

Wenn auch oft lange Formeln auftreten, ist die ganze Darstellung übersichtlich gestaltet, mit vielen Erläuterungen, Hinweisen auf frühere Arbeiten, Vergleichen mit anderen Untersuchungen und Theorien, auch Kurven- und Zahlenmaterial versehen, und beschließt mit einem ausführlichem Literaturverzeichnis. D. Morgenstern.

Castoldi, Luigi. Rappresentazioni equivalenti di moti stazionari di fluidi incompressibili con congruenza normale di linee di corrente. In particolare: moti „quasi-Euleriani“. Rend. Sem. Fac. Sci. Univ. Cagliari 25 (1955), 37-43.

The author begins by remarking that while a complex-lamellar field may always be expressed in the form $v = f \text{ grad } g$, it is sometimes more convenient to use the less specific form

$$v = h \text{ grad } \psi + \text{grad } F(h, \psi).$$

He gives conditions for choice of h , ψ , and F such that the second form reduces to the first, and he notes involuntary transformations for the first form. He calls „quasi-Eulerian“ a solenoidal complex-lamellar velocity field which approaches a limit at ∞ and gives some examples of such fields. C. Truesdell (Bloomington, Ind.).

Yeh, G. C. K.; Martinek, J.; and Ludford, G. S. S. The potentials due to certain singularities in the presence of a fixed sphere. J. Soc. Indust. Appl. Math. 3 (1955), 142-152.

The sphere theorem of the authors [Proc. Cambridge Philos. Soc. 51 (1955), 389-393; MR 17, 678] is applied to find the hydrodynamic images of the following systems interior to a sphere: source and sink, doublet, circular vortex ring (which lies on a concentric sphere). The same systems external to the sphere are also discussed.

L. M. Milne-Thomson (Greenwich).

Gheorghitǎ, Șt. I. Sur les mouvements fluides rotatoires. Acad. Repub. Pop. Romine. Bul. Ști. Sect. Ști. Mat. Fiz. 7 (1955), 393-399. (Romanian. Russian and French summaries)

Stream functions are constructed for plane incompressible flows about rotating cylindrical objects with the

following cross sections: a circle with congruent radial fins at equal intervals about its circumference; the same object with equal gaps between fins and circular core; a circular sector; equally spaced congruent circular sectors joined at their vertices; and the intersections of two circles of equal radii. Some of the published results are erroneous but can easily be corrected.

J. H. Giese (Aberdeen, Md.).

Kočina, I. N. On a class of vortical motions of an ideal incompressible fluid. *Prikl. Mat. Meh.* 19 (1955), 756-759. (Russian)

A simple proof is given of a remarkable unpublished theorem of Caplygin as follows: "Three-dimensional flow in which the vertical velocity component $w=0$, and $\partial u/\partial y - \partial v/\partial x = 0$ in any plane parallel to the plane $z=0$, can be obtained from any steady plane parallel two-dimensional flow, given in the plane $z=0$, by turning the velocity vector (in the plane) through the angle $\theta(z)$, constant for the whole plane, where $\theta(z)$ is an arbitrary function of z . In going from plane to plane along a vertical the fluid speed does not change".

The author proceeds to examine some cases of gravity waves of small amplitude and given form on the surface of water of depth h flowing in the above manner with constant speed c . Relations between speed and wave-length are obtained for some particular forms of $\theta(z)$ but no physical interpretations are offered.

L. M. Milne-Thomson.

Haskind, M. D. Unsteady gliding on an undulating surface of a heavy fluid. *Prikl. Mat. Meh.* 19 (1955), 331-342. (Russian)

The author treats the (two-dimensional) problem of a contour, of projected length $2a$ and only slight curvature, gliding with average forward speed u on the surface of a fluid with infinitesimal waves of frequency σ . The problem of a glider on still water has been treated earlier by several persons. Since the problem is linearized here, the author is able to separate out equations expressing the effect of the waves. These are treated by methods from the theory of functions of a complex variable (two imaginary units are used) and lead to an infinite system of linear equations whose solvability is studied for limiting values of the parameters $\mu = ga/u^2$ and $\omega = \sigma a/u$. Formulas are derived for vertical force and moment on the contour as a result of the waves.

J. V. Wehausen.

Moiseev, N. N. On a problem of the theory of waves on the surface of a bounded volume of fluid. *Prikl. Mat. Meh.* 19 (1955), 343-347. (Russian)

The author treats two-dimensionally the problem of the wave motion of a fluid with a free surface in a container with an opening where the pressure is given by either (i) $p = -\rho gy + \rho c \cos \omega t$ or (ii) $p = -\rho gy$. His results are expressed in the theorem: The frequencies of the free oscillations for case (ii) are given by $\omega_n = (\lambda_n g)^{1/2}$, where λ_n are the eigenvalues of a certain integral equation; for case (i) forced oscillations are possible for any $\omega \neq \omega_n$. A special case is carried through and a generalization of case (i) is considered.

J. V. Wehausen (Berkeley, Calif.).

Palm, Enok. On the formation of surface waves in a fluid flowing over a corrugated bed and on the development of mountain waves. *Astrophys. Norvegica* 5 (1953), 61-130.

In determining the surface waves resulting when fluid with a free surface flows steadily about or over an obstruction, the solution is not unique without additional

restrictions besides the linearized free-surface condition. Uniqueness may be achieved by either requiring that the wave motion vanish far ahead of the obstruction (Kelvin) or by introducing during part of the solution a fictitious viscosity of no physical significance (Rayleigh). The author considers an initial-value problem in which the free surface is initially at rest and an irregular bottom, expressible by a Fourier integral, starts moving at $t=0$ with constant velocity. He shows that when $t \rightarrow \infty$ the solution for the free surface approaches that obtained in the steady-state problem by using either of the devices mentioned above [see also Stoker, *Comm. Pure Appl. Math.* 6 (1953), 471-481; MR 15, 660]. The author treats similarly an analogous meteorological problem arising in the undulating flow of air in the lee of an obstruction, like a mountain.

J. V. Wehausen (Berkeley, Calif.).

Kiselev, A. A. On the unsteady flow of a viscous fluid with external forces. Translated by Morris D. Friedman, 2 Pine St., West Concord, Mass., 1955. 6 pp.

Translation of *Dokl. Akad. Nauk SSSR (N.S.)* 100 (1955), 871-874; MR 16, 1061.

Kiselev, A. A. Solution of the linearized equations of nonsteady viscous incompressible fluid flow in a bounded region. Translated by Morris D. Friedman, 2 Pine St., West Concord, Mass., 1955. 7 pp. \$3.50.

Translation of *Dokl. Akad. Nauk SSSR (N.S.)* 101 (1955), 43-46; MR 16, 1170.

Wilkinson, J. A note on the Oseen approximation for a paraboloid in a uniform stream parallel to its axis. *Quart. J. Mech. Appl. Math.* 8 (1955), 415-421.

The flow of an incompressible viscous fluid past a paraboloid of elliptic section is calculated by Oseen approximation. The limiting case of a flat plate with parabolic leading edge is investigated in detail. It shows that in this three-dimensional flow the skin friction on the plate is also in the direction of the motion. By comparison with results obtained by other methods, the author concludes that the edge effects are small and that the boundary-layer approximation can be justified.

Y. H. Kuo (Ithaca, N.Y.).

Fil'čakov, P. F. On the work on filtration of the Institute of Mathematics of the Academy of Sciences of the Ukrainian SSR using large machines. *Ukrain. Mat. Ž.* 4 (1952), 111-119. (Russian)

Ibrahim, Ali A. K. Equation of motion in circles about an axis for non-Newtonian liquids. *Z. Angew. Math. Mech.* 35 (1955), 463-464.

Despite the title, the author appears to be dealing with linearized visco-elasticity in the plane case. He calculates the curl of the acceleration at length and thus derives an equation satisfied by the vorticity. In the latter part of the paper, which concerns a simple vortex, the reviewer cannot follow the details.

C. Truesdell.

Rivlin, R. S. Solution of some problems in the exact theory of visco-elasticity. *J. Rational Mech. Anal.* 5 (1956), 179-188.

This paper deals with isotropic fluid materials defined by the assumption that the stress components are polynomials in the components of the gradients of the velocity and its material time derivatives up to a certain order.

The general form of such constitutive equations was studied previously by the author and Ericksen [same J. 4 (1955), 323-425, 681-702; MR 16, 881; 17, 210]. On the basis of the results obtained in these papers the author investigates some simple types of steady flow for incompressible materials: rectilinear shearing flow, torsional flow between two parallel discs, helical flow between concentric cylinders. It is assumed that body forces are absent and, in the case of torsional flow, that centrifugal forces can be neglected. The results are compared with those of more special non-linear theories of fluids.

W. Noll (Los Angeles, Calif.).

Zaat, J. A.; van Spiegel, E.; and Timman, R. The three-dimensional laminar boundary layer flow about a yawed ellipsoid at zero incidence. Nat. Luchtvaartlab. Amsterdam. Rep. F 165 (1955), 16 pp. (1 insert).

Timman's method for calculation of three-dimensional boundary layers [same Rep. no. F66 (1950); MR 12, 871] is reviewed briefly. In it, the boundary-layer equations for a smooth surface are expressed in potential-streamline coordinates, and two coupled, first-order momentum equations are set up. The case calculated here is that of an ellipsoid of three unequal axes (3, 1, and 0.15) placed in a stream of incompressible fluid so that that stream velocity vector lies in the plane of the two longer axes and bisects the angle between them, simulating a yawed wing of elliptical planform at zero incidence. Numerical results giving the details of the flow are presented graphically and in tables. W. R. Sears (Ithaca, N.Y.).

Gvozdkov, N. N. Nonsteady motion of a viscous fluid in the boundary layer. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. 15 (1955), 99-105. (Russian)

Noting that a boundary layer creates vorticity, the author endeavors to develop a boundary-layer theory for non-steady motion based on vortex theory. Starting with the vortex transport equation, the author recalls that in the initial instants in accelerated motion the convective terms may be neglected, and that the simplified (now linear) equation can be solved exactly for single vortex in the origin. The boundary layer on two sides of a body is replaced by two counter-rotating systems of vortices. The distribution of intensities is selected so as to satisfy the no-slip condition. The case of an infinitely thin plate of length l started impulsively is treated in detail, on the assumption that each of the two systems of vortices affects the flow on its side only. This allows the author to construct an expression for the logarithmic potential, and to derive expressions for the velocity components. These are given in closed form as triple integrals. An expression for skin friction (involving a singular integral) is also derived. The author states that the same idea can be applied to uniform acceleration, and to arbitrary shape by the use of conformal mapping.

The utility of the author's idea cannot be judged in the absence of more detailed results. No comparison of the case treated with the well-known Blasius solution [Z. Math. Phys. 56 (1908), 1-37] or experiment is given.

J. Kestin (Providence, R.I.).

Gvozdkov, N. N. On approximate equations of motion in a thin layer of a viscous compressible fluid with heat transfer. Voronezh. Gos. Univ. Trudy. Fiz.-Mat. Sb. 27 (1954), 14-19. (Russian)

The author writes down equations for plane, steady motion of compressible fluid and renders them dimension-

less (without carefully explaining the meaning of symbols). Assuming $\epsilon = \delta/l$ (mean boundary-layer thickness to its length) to be small, the author considers two cases: (a) motion when parameter $A = U\delta^2/c_\mu g T_0$ (subscript "0" denotes reference quantities) is small and Reynolds number not too large; it is assumed here that $1/Re = O(\epsilon)$ and $A = O(\epsilon)$; (b) motion with $A = O(1)$ and large Reynolds numbers; it is assumed here that $1/Re = O(\epsilon^2)$ and $A = O(\epsilon^0)$. For both cases suitably simplified equations are written down (not derived). Solutions are assumed to be in the form of series in ϵ . Equations involving successive coefficients of powers of ϵ in the respective expansions are written down. The author claims that such a scheme will facilitate the calculation of more exact approximations in gas dynamics. J. Kestin (Providence, R.I.).

Kuo, Y. H. Viscous flow along a flat plate moving at high supersonic speeds. J. Aero. Sci. 23 (1956), 125-136.

L'auteur étudie l'écoulement plan, permanent, supersonique d'un fluide visqueux compressible le long d'une demi-plaque. Le point de départ est l'équation que vérifie la fonction de courant. Une solution approchée est construite par la méthode de Lighthill [Phil. Mag. (7) 40 (1949), 1179-1201; MR 11, 518] qui consiste à exprimer la fonction inconnue et les variables au moyen de paramètres auxiliaires. La viscosité prise en considération au voisinage de la plaque est négligée au loin; les deux solutions coïncident au second ordre près le long d'une certaine ligne qui constitue la frontière de la couche limite; x désignant la distance au bord d'attaque, l'épaisseur de la couche limite augmente comme $x^{3/4}$. La présence de la couche limite entraîne l'existence d'une onde de choc attachée au bord d'attaque pour les grandes valeurs du nombre de Mach; le coefficient de frottement commence par croître avec x , il décroît ensuite. H. Cabannes.

★ Obukhov, A. M. On the effect of weak atmospheric inhomogeneities on sound and light propagation. Translated by Morris D. Friedman, 2 Pine St., West Concord, Mass., 1955. 16 pp. \$8.00.
Translated from Izv. Akad. Nauk SSR. Ser. Geofiz. 1953, 155-165; MR 15, 1002.

Roth, H. Pressure distribution on a wall under impact of a subsonic gas jet. Acta Phys. Austriaca 10 (1956), 142-148.

Il s'agit d'une application de la méthode classique de Chaplygin pour l'étude des écoulements plans compressibles subsoniques. L'auteur effectue une application numérique lorsque le nombre de Mach du jet est $M=0,826$ et montre ainsi que dans des conditions comparables, la chute de pression le long du mur est d'autant plus lente que le nombre de Mach de l'écoulement est plus élevé.

P. Germain (Paris).

Kusukawa, Ken-ichi. On the subsonic flow of a compressible fluid past an axisymmetric moderately thick body. J. Phys. Soc. Japan 10 (1955), 1093-1101.

In an earlier paper [same J. 9 (1954), 605-610; MR 16, 301] the author worked out an approximate theory for subsonic flow past slender bodies. Here it is extended to thicker obstacles. The new equations, in hodograph space, are not so simple because the radial coordinate y still appears. It is proposed to use first the earlier approximation and to substitute the resulting first approximation to y into the present equations, which can then be solved. As examples, the flows past prolate spheroids of thickness

ratios 0.1, 0.9 and 1.0 (sphere), at several Mach numbers, are treated.

W. R. Sears (Ithaca, N.Y.).

Miles, John W. On the sonic drag of a slender body. *J. Aero. Sci.* 23 (1956), 146-154.

L'auteur reprend l'analyse classique de la théorie des corps élancés selon laquelle le potentiel, au voisinage de l'obstacle, est dans chaque section normale au vent, dirigé suivant l'axe des x , une fonction harmonique. Les conditions aux limites déterminent cette fonction à une constante additive près variable avec x . Cette constante joue un rôle essentiel dans l'évaluation de la traînée. Dans une première partie est établie la formule donnant l'expression de cette traînée par application du théorème des quantités de mouvement. Les intégrales portant sur les parties de la surface de contrôle situées près de l'obstacle s'évaluent à l'aide de la théorie des corps élancés, mais il reste une intégrale prise sur la surface sonique, où une telle approximation ne peut plus être valable. C'est pour évaluer une telle intégrale que l'auteur utilise le résultat de Guderley et Yoshihara [*Quart. Appl. Math.* 8 (1951), 333-339; *MR* 12, 553] donnant l'allure asymptotique à l'infini d'un écoulement sonique de révolution, sous la forme d'une fonction singulière en un point de l'axe des x . Des conditions de continuité pour le raccord des deux solutions laissent inconnues en définitive, dans des cas assez larges, l'abscisse du point sonique et celle du point singulier. Sans entrer dans le détail de l'analyse, signalons que cette singularité est placée au centre des sources schématisant l'obstacle jusqu'à la frontière transsonique. L'auteur applique enfin sa théorie aux obstructes présentant un épaulement et aux fuselages ayant une pointe arrière. Les résultats numériques montrent que dans le premier cas, la traînée a l'ordre de grandeur prévu par la théorie supersonique des corps élancés, alors que dans le second elle est considérablement plus faible.

P. Germain.

Mangler, K. W. Calculation of the pressure distribution over a wing at sonic speeds. *Aero. Res. Council, Rep. and Memo. no. 2888* (1951), 55 pp. (1955).

This is the final published version of a report written in 1951 and widely distributed since then. In it the Munk-Jones theory of very slender wings is applied to problems of tapered and sweptback wings. This theory is applicable whenever $|1-M^2|A^2$ is small compared to 1, where M denotes the Mach number of flight and A the aspect ratio of the planform. Earlier work by Robinson [*Coll. Aero. Cranfield. Rep. no. 41* (1950); *Aero. Quart.* 4 (1952), 69-82; *MR* 12, 452; 14, 219] is here extended, and corrections due to Mirels [*NACA Tech. Note no. 3105* (1954); *MR* 15, 839] are incorporated.

W. R. Sears (Ithaca, N.Y.).

v. Krzywoblocki, M. Z.; and Shinosaki, G. On drag of some bodies in free molecule flow. *Acta Phys. Austriaca* 10 (1956), 34-53.

The second approximations to the drags of a paraboloid of revolution, and of a circular flat plate at angle of obstacle, are obtained on free molecule theory, using the assumptions (alternatively) of diffuse and specular reflexion.

M. J. Lighthill (Manchester).

Davies, C. N.; and Peetz, C. V. Impingement of particles on a transverse cylinder. *Proc. Roy. Soc. London. Ser. A.* 234 (1956), 269-295.

When particles suspended in a stream sweep over an object, some of them will collide with it, but others will miss. This paper gives approximate methods for calcula-

ting the number of particles striking a circular cylinder, taking into account the size of the particles. The investigation was carried out under different assumptions. First, the flow with respect to the cylinder is irrotational but viscous with respect to the suspended particles. Secondly, the viscous effects in the case of thin cylinders and low Reynolds numbers are included and the same problem studied. The results show that, for given particle size, an increase of the radius of the cylinder increases the collision efficiency; and, for given cylinder, a decrease of the particle size increases the collision efficiency.

Y. H. Kuo (Ithaca, N.Y.).

Ščerbakov, L. M.; and Bolotin, A. S. On the dependence of surface tension on the radius of a drop. *Kišinev. Gos. Univ. Uč. Zap.* 11 (1954), 153-156. (Russian)

MacLean, William R. Zur Theorie der Wellenausbreitung in nichthomogenen Medien. *Z. Physik* 143 (1955), 331-339.

The vector approach to the theory of electromagnetic waves in a non-homogeneous unsymmetrical medium presents insuperable difficulties in that there is no vector-potential representation of the electromagnetic field. One has therefore to use scalar wave theory, i.e. the theory of sound waves. The standard method of dealing with diffraction problems in the theory of sound waves involves the use of Green's function for the whole of space, and the Green's function is complicated when the medium is non-homogeneous. The object of the paper is to obtain a comparatively simple approximation to the Green's function.

If the medium has variable density ρ and variable adiabatic elastic constant σ , "monochromatic" acoustic waves in the medium satisfy the equations

$$\frac{1}{\rho} \text{grad } p = -\omega^2 q, \quad p = -\sigma \text{div } q,$$

where p is the pressure, q the displacement, from which it follows that

$$\eta \beta \text{div} \left(\frac{1}{\eta \beta} \text{grad } p \right) + \beta^2 p = 0,$$

where $\eta = (\sigma \rho)^{1/2}$, $\beta = \omega/c$, and c is the wave-velocity $(\sigma/\rho)^{1/2}$.

The author makes the point that β behaves like a refractive index, so that variations in β (or c) cause refractions; but in refraction, η plays no part. He therefore states that, if he replaces the real value of η by an artificial one, he will not alter the refractive powers of the medium, only its reflective powers. His aim is then to get a simple formula for the Green's function by a suitable choice of η . There is not space here to give any details of the method of choosing η or of the final results.

E. T. Copson (St. Andrews).

Proudman, J. The effect of friction on a progressive wave of tide and surge in an estuary. *Proc. Roy. Soc. London. Ser. A.* 233 (1955), 407-418.

This is a second paper on the distribution of a combination of tide and surge along an estuary. In the first paper [same *Proc.* 231 (1955), 8-24; *MR* 17, 102] the author treated the cases of standing oscillations, a single progressive wave and two waves; in addition the effect of the non-linear inertial term was included. However, the results obtained were subject to a restriction on the ratio of the product of the wave height and the distance from the sea to the square of the depth of the water, and

were only applicable for places near the mouth of the estuary. In the present paper the case of a single progressive wave only is treated and the non-linear term is neglected, but the above restriction is now removed, so that the formulas are valid for any height of surge or tide at the mouth, and for any distance from the sea. Friction is assumed to be proportional to the square of the speed of the current, and the effect of the contraction of the estuary is treated by means of Green's method. It is found that if the surge at the mouth of the estuary is greater than a certain fraction of the tidal range there, then the surge at any point up the estuary is less at time of high water than at low water. *M. H. Rogers* (Urbana, Ill.).

Crease, J. Propagation of long waves due to atmospheric disturbances on a rotating sea. *Proc. Roy. Soc. London. Ser. A.* 233 (1956), 556-569.

This is an investigation of the development and propagation of surges on a flat rotating sea. The surges are initiated by air pressure gradients or wind stresses, and it is assumed that the force is stationary, of constant amplitude and is suddenly applied and maintained over one half of the sea. The author considers the case when conditions are uniform in one horizontal direction and the force is applied at right angles to this direction. The equations of motion are integrated from the surface to the bottom of the sea and solutions for the mean horizontal velocities are obtained. The surface elevation and the velocity field are examined in some detail, and a study is also made of conditions after a long time has elapsed. It is found that the effect of the discontinuity in the applied force at the edge of the generating area travels into the undisturbed sea and the generating area itself with the maximum group velocity of the system. When a long time has elapsed an inertial oscillation with an amplitude decreasing with distance from the generating area is attained, but in addition to this there is a component of the transverse velocity which balances the forcing function.

The case of a surge due to a travelling disturbance is examined briefly; when the edge of the generating area is advancing with the maximum group velocity of the system, a surge with an elevation of ever-increasing height is produced. Finally the author considers the effect of a barrier at right angles to the direction of propagation.

M. H. Rogers (Urbana, Ill.).

See also: Richards, p. 761.

Elasticity, Plasticity

★ **Sokolnikoff, I. S.** *Mathematical theory of elasticity*. 2d ed. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1956. xi+476 pp. \$9.50.

In the preface to the first edition [1946] of his book the author announced a second volume, devoted to the plane theory of elasticity and shell theory. Instead he chose to modify the book in its second edition to include a wider aspect than would have been covered in the originally planned two volumes with the exception that shell theory has been excluded, due to lack of space.

The teaching of classical elasticity has been greatly influenced by the first edition, on the one hand by the introduction of orthogonal tensors, on the other hand by the presentation of powerful methods of solution, initiated in Russia by G. V. Kolosoff and N. I. Muskhelishvili whose work up to that time was relatively unknown due to lack of translations.

The present edition, in particular, pursues the second object and in this respect renders an invaluable service in view of the fact that over the past ten years Russian mathematicians have continued to produce important work in the fundamental as well as applied fields of the subject.

The first three chapters (90 pages), giving the fundamentals of the theory, have remained largely unchanged, apart from a short historical sketch, remarks on Mohr's diagram and on the existence of solutions, indicating that the presentation has withstood the test of time. Chapter IV (158 pages), dealing with the extension, torsion and flexure of beams, has been slightly extended to include flexure of beams with semicircular cross-section, multiply-connected crosssections, non-homogeneous beams and deformation of cylinders by loads on the side surfaces.

Chapters V (80 pages) and VI (49 pages) are new. The former presents a summary of work in the plane theory of elasticity using complex-function theory. The methods of solution, originating in Russia, are illustrated by applications; special consideration is given to Schwarz's alternating method for the case of multiply connected regions. The use of these methods for anisotropic materials and lateral deflection of plates is only mentioned in passing.

Chapter VI on the three-dimensional problems describes the important developments in this field since the last edition of A. E. H. Love's book in 1926. The use of harmonic functions for solving Navier's equations forms the bulk of this chapter. It also deals with Betti's method of integration and gives an introduction to thermoelastic, dynamic and wave theory.

Chapter VII (89 pages) bears the same title as Chapter V of the first edition, but has been rewritten and considerably extended. It establishes the general theorems on strain energy in deformed bodies and illustrates their application. Approximate methods of solution due to Ritz, Galerkin, Kantorovich and others are developed and applied to particular problems. The methods of collocation least squares, finite differences and relaxation are also introduced.

J. R. M. Radok (Berkeley, Calif.).

★ **Lur'e, A. I.** *Prostranstvennye zadachi teorii uprugosti. [Spatial problems of the theory of elasticity]*. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1955. 491 pp. 17.60 rubles.

This is a comprehensive study of the three-dimensional theory of elasticity by a leading Russian mathematician in this field. The book is built around the author's own published and unpublished material and contains detailed references to work inside and outside Russia as late as 1953.

Chapter I (62 pages) establishes the basic equations in curvilinear coordinates without use of general tensors. The general solution of these equations in terms of harmonic functions is discussed in detail. Special consideration is given to body forces, deduced from a potential, and to thermal effects. Chapter II. (14 pages) deals with the unbounded and semi-unbounded space, subject to internal and external concentrated and distributed forces. Singular solutions are discussed in general and the solution of the problem of an elastic cone, loaded at its vertex, is deduced as a generalization of the problem of a concentrated force acting on the half-space.

Chapter III (54 pages) is concerned with the solution of problems of elastic layers by a formal method using

algebraic equations in differential operators. The method is applied to the problems of compression by direct forces on the free surfaces, of bending and of thermal stressing. Chapter IV (51 pages) studies problems of thick plates by the method of the preceding chapter. The solution is obtained in two parts, the "homogeneous" which satisfies the edge conditions and the "non-homogeneous", satisfying the loading conditions on the free faces. The particular cases considered are the circular plate for various edge and loading conditions and thermal stresses due to linear variation of temperature through the plate thickness.

Chapter V (75 pages) deals with the contact problem. Circular and elliptical areas of contact are solved as special cases. Chapter VI studies the deformation of symmetrically loaded spheres, thick spherical shells and the ellipsoidal cavity. Chapter VII (60 pages) deals with the circular cylinder in bending and under loads on its side surface. Chapter VIII (45 pages) returns to the problem of the elastic sphere for more general boundary conditions and solves several particular cases for given surface tractions and displacements.

Each chapter is followed by a survey of the literature relating to the presented results. J. R. M. Radok.

★ Pailloux, H. *Elasticité*. Mémor. Sci. Math., no. 132. Gauthier-Villars, Paris, 1956. 91 pp. 1000 francs. Introductory treatment of a number of topics in the theory of elasticity. E. Reissner (Cambridge, Mass.).

Green, A. E. *Hypo-elasticity and plasticity*. Proc. Roy. Soc. London. Ser. A. 234 (1956), 46-59.

Hypo-elasticity is a new theory of materials whose response is specified in terms of rates. It differs from older theories of a similar kind in that its equations, while reducing to those of the classical linearized theory of elasticity under the assumptions usual in formulating that theory, are of a form admissible for deformations of any magnitude and speed. Previous work in hypo-elasticity has concentrated mainly on phenomena of elastic type, although it has gone as far as predicting phenomena indicating yield or rupture as the limit of elastic response.

In this paper the author shows that several of the theories usually proposed for plastic bodies are included as special cases or limit cases within hypo-elasticity. The work differs from older analysis by Prager [Proc. 5th Internat. Congress Appl. Mech., Cambridge, Mass., 1938, Wiley, New York, 1939, pp. 234-237, and later papers] not only in its greater generality but also in being valid for large deformations. The author restricts attention to incompressible bodies, for which he proposes equations somewhat different from those given by the reviewer. As in most theories of strain hardening, the author proposes to use different constitutive equations for loading and for unloading, and he finds conditions on the hypo-elastic coefficient functions in order that the two sets be compatible.

The author solves exactly the problem of simple extension for an incompressible body in a particular type of motion. He finds that the stress system is not a state of simple tension and is not homogeneous; it depends upon the strain and the rate of strain. If in the exact solution so obtained the density is allowed to approach zero, stress-strain relations emerge in the limit and are of the same form as results in a quasi-static treatment in which inertia is neglected and cross-stresses are annulled. For a special

case, the author obtains approximately a stress which tends monotonically to a finite limit as strain increases, implying a gradual transition from the elastic to the plastic range. For the same special equations, the time derivative of the stress intensity reaches a certain value, suggesting a yield of the Maxwell-Mises type. [A similar observation was made independently by T. Y. Thomas, Proc. Nat. Acad. Sci. U.S.A. 41 (1955), 908-910; MR 17, 321.] The author remarks that the yield value of the stress intensity is reached only at infinite strain.

Similar phenomena are found by the author for simple shear. [Here, for a compressible body, the reviewer has shown in a paper to appear in J. Appl. Phys. that the implied plastic yield may never occur for real strains, and that if it does occur a primary or hypo-elastic yield, consisting in a maximum of stress at a finite strain, always occurs first and at a stress exceeding the limit stress.]

The author obtains solutions representing expansion of a cylindrical tube and torsion of a circular cylinder. For the former, the inertia of the material is not neglected. The author shows that when certain approximations are made, his solutions reduce to those obtained in some works on plasticity. [The reviewer conjectures that because of the author's expansion in powers of a material coefficient, a modification like that mentioned above for simple shear will be necessary.] C. A. Truesdell.

Jindra, Friedrich. *Eindimensionale Probleme bei einem nichtlinearen Elastizitätsgesetz*. Z. Angew. Math. Phys. 6 (1955), 345-355.

The author considers a stress-strain law, valid for infinitesimal strain, which differs from the usual law of linear elasticity by allowing the elastic moduli to depend in a very special way on the first and second invariant of the stress tensor. He investigates homogeneous strain, inflation of a thick cylindrical tube and of a spherical shell. W. Noll (Los Angeles, Calif.).

Ericksen, J. L. *Stress deformation relations for solids*. Canad. J. Phys. 34 (1956), 226-227.

This note concerns the form of the stress-strain relation when the stress t_{ik} is assumed to be a function of the displacement gradients $x_{i,k}$ only. Under the very weak assumption that in any neighborhood of any deformation there is another deformation which cannot be attained in a motion in which the stresses do no work it is proved that the stress-strain relation necessarily reduces to the form

$$t_{ij} = \rho \phi x_{i,k} \partial \psi / \partial x_{j,k},$$

where ρ denotes the density and where ϕ and ψ are appropriate functions of the $x_{i,k}$. In the special case when ϕ depends only on ψ the existence of a strain-energy function can be inferred. W. Noll.

Stoppelli, Francesco. *Sulla sviluppabilità in serie di potenze di un parametro delle soluzioni delle equazioni dell'Elastostatica isoterma*. Ricerche Mat. 4 (1955), 58-73.

In a previous paper [Ricerche Mat. 3 (1954), 247-267; MR 17, 554], the author has proved a general theorem of existence and uniqueness for the classical theory of finite elastic strain. This work was motivated by a formal power series solution due to Signorini, but made no use of it. The extrinsic forces and surface loads were written with a multiplicative factor θ , and existence and uniqueness were proved subject to the assumptions that the loads do

not possess an axis of equilibrium and that $|\theta|$ is sufficiently small.

The present paper proves that under the same assumptions regarding the loads, the solution is analytic in θ . Thus Signorini's perturbation series and some of the more recent perturbation methods of finite elasticity are justified. The proof uses apparatus similar to that in the previous paper. The problem is reduced in such a way that the essential step is the proof that in the ordinary linear elasticity, if the loads are analytic functions of θ the solution for the displacement which vanishes and is strictly irrotational at an assigned point is also an analytic function of θ .

C. A. Truesdell.

Caprioli, Luigi. Su un criterio per l'esistenza dell'energia di deformazione. *Boll. Un. Mat. Ital.* (3) 10 (1955), 481-483.

It is shown that the existence of a strain-energy function for a linear or non-linear elastic material is a consequence of the following assumptions: 1) the stress tensor depends exclusively on the strain tensor, and 2) the work necessary to produce any deformation starting from the stress-free state is non-negative. Possible applications of the underlying mathematical lemma to electromagnetic theory are indicated.

W. Prager.

Tagliacozzo, Carlo. Sulla dilatazione cubica totale di un solido omogeneo ed isotropo in coazione elastica. *Univ. e Politec. Torino. Rend. Sem. Mat.* 14 (1954-55), 269-276.

Betti's theorem enables one to calculate the mean cubical dilatation when the loads are known. The author asserts that he has proved in an earlier note [*Revista de Engenharia Mackenzie*, Sao Paulo No. 92 (1946)] that the displacements arising from an assigned deformation are the same as those arising from a certain distribution of load. Using this result, the author derives a formula asserting that the mean dilatation equals the mean impressed dilatation. This generalizes a known theorem on the mean thermoelastic dilatation.

C. Truesdell.

Lang, H. A. The affine transformation for orthotropic plane-stress and plane-strain problems. *J. Appl. Mech.* 23 (1956), 1-6.

Solutions of two-dimensional problems for a restricted type of orthotropic material are found from corresponding two-dimensional isotropic problems by an affine transformation. [The solution of the problem discussed by the author is known for a general anisotropic body. The special method proposed by the author, which gives results only for a restricted type of orthotropy, appears to have no particular virtue.]

A. E. Green.

Iacovache, Maria. L'application des fonctions monogènes au sens de Feodorov à la théorie de l'élasticité des corps à isotropie transverse. *Rev. Univ. "C. I. Parhon" Politehn. București. Ser. Ști. Nat.* 1 (1952), no. 1, 58-60. (Romanian. Russian and French summaries)

The author shows that the general solution of the equations of plane linear elasticity for bodies with transverse isotropy is an arbitrary analytic function of a certain type of hypercomplex variable. [Cf. Burgatti, *Boll. Un. Mat. Ital.* 1 (1922), 8-12; Aymerich, *Rend. Sem. Fac. Sci. Univ. Cagliari* 17 (1947), 1-12; MR 10, 534.]

C. Truesdell (Bloomington, Ind.).

Ionescu-Cazimir, Viorica. Sur les équations de l'équilibre thermo-élastique plan. *Rev. Univ. "C. I. Parhon" Politehn. București. Ser. Ști. Nat.* 1 (1952), no. 1, 55-57. (Romanian. Russian and French summaries)

In plane linear thermoelasticity Airy's function satisfies

$$\nabla^4(\nabla^2 - \nu \frac{\partial}{\partial t})\Omega = 0,$$

where ν is a thermoelastic coefficient.

C. Truesdell.

Sneddon, Ian. N. The stress produced by a pulse of pressure moving along the surface of a semi-infinite solid. *Rend. Circ. Mat. Palermo* (2) 1 (1952), 57-62.

The problem is considered for plane strain. With the usual notation, the displacement components are taken in the form

$$u = \frac{\partial \varphi}{\partial x} + \frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \varphi}{\partial y} - \frac{\partial \psi}{\partial x},$$

and it is shown that φ and ψ satisfy wave equations. Using complex functions, combinations of special solutions of the type $f(x - vt \pm iky)$ are shown to satisfy the boundary conditions. Representation of the pressure pulse by a Fourier integral is applied for specific examples. A similar solution for a pulse of shear traction is also given.

E. H. Lee (Providence, R.I.).

Sadowsky, Michael. Stress concentration caused by multiple punches and cracks. *J. Appl. Mech.* 23 (1956), 80-84.

The plane-strain problem for a half-plane is treated by complex-variable methods, the boundary conditions being $\tau_{xy} = 0$ on the entire x -axis, $\sigma_y = 0$ for free parts, $v = \text{constant}$ for punched parts (punch and crack problems are equivalent). Detailed results are given for a double punch. The solution was obtained by mapping the x -axis onto a rectangle so that its corners correspond to the corners of the punch.

R. C. T. Smith (Cambridge, Mass.).

Conway, H. D. The nonlinear bending of thin circular rods. *J. Appl. Mech.* 23 (1956), 7-10.

The subject of the non-linear bending of thin rods has received new attention of late, solutions of various specific problems having been given by H. J. Barten [*Quart. Appl. Math.* 2 (1944), 168-171; MR 6, 83], K. E. Bisshopp and D. C. Drucker [*ibid.* 3 (1945), 272-275; MR 7, 143], F. Hymans [*Trans. A.S.M.E.* 68 (1946), A-223-A-230] and H. D. Conway [*Philos. Mag.* (7) 38 (1947), 905-911; MR 9, 638]. Most of the papers have been confined to rods bent by concentrated forces or couples because of the relative simplicity of such problems. More complicated problems of uniformly distributed loading have been treated by F. V. Rohde [*Quart. Appl. Math.* 11 (1953), 337-338; MR 15, 75] and C. Truesdell [*Proc. 1st Midwestern Confer. Solid Mech.*, 1953, Univ. of Illinois, pp. 52-55]. The present paper deals with the bending of initially circular rods by concentrated forces. Two examples are discussed using the Bernoulli-Euler equation, which states that the change of curvature is proportional to the bending moment. Numerical results and graphs are presented, illustrating the difference between the linear and the non-linear theory.

R. Gran Olsson (Trondheim).

Bahtin, I. A.; and Krasnosel'skiĭ, M. A. On the problem of longitudinal bending of a rod of variable stiffness. Dokl. Akad. Nauk SSSR (N.S.) 105 (1955), 621-624. (Russian)

The longitudinal displacement $y=y(s)$ of a bent rod at a point whose arc length from one end is s ($0 \leq s \leq 1$) satisfies the equations

$$(1) \quad y'' = -P\varrho(s)y(1-y'^2)^{\frac{1}{2}}, \quad y(0)=y(1)=0,$$

where $\varrho(s)$ is the stiffness and P is the load. The load P_0 is critical if for arbitrary $\varepsilon > 0$, $\delta > 0$ there exists for some P with $|P-P_0| < \delta$ a nontrivial solution of problem (1) satisfying $|y(s)| < \varepsilon$. To find the critical values of P the authors replace (1) by the equivalent integral equation

$$(2) \quad \varphi(s) = P\varrho(s) \int_0^1 G(s, t) \varphi(t) dt \left(1 - \left[\int_0^1 G_s'(s, t) \varphi(t) dt \right]^2 \right)^{\frac{1}{2}},$$

where $\varphi(s) = -y''(s)$ and the symmetric $G(s, t) = s(1-t)$ for $s \leq t$. Application of results and methods developed by Krasnosel'skiĭ [Uspehi Mat. Nauk (N.S.) 9 (1954), no. 3(61), 57-114; MR 17, 769] gives the result that the critical loads are exactly the eigenvalues P_k of the linearized problem (3) $y'' = -P\varrho(s)y$, $y(0)=y(1)=0$, and that to each P_k there corresponds an interval of P -values with P_k as left endpoint for which (1) has a nontrivial solution $y(s)$ which together with $y''(s)$ tends to 0 as $P \rightarrow P_k$. If (1) is replaced by

$$(4) \quad \frac{d^2 y}{dx^2} = -P\varrho(x)y \left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{-3/2}, \quad y(0)=y(1)=0,$$

where the abscissa x replaces the arc length s , then the P -values for which (4) has such nontrivial solutions fill intervals with the P_k as right endpoints. Since this is physically implausible it is concluded that (1) is the more correct formulation of the problem. *M. Golomb.*

Conti, Lorenzo. Delle lastre e travi-parete rettangolari di spessore variabile. Rend. Sem. Mat. Univ. Padova 24 (1955), 346-352.

This note considers the bending of thin plates when the thickness of the plate varies in one direction only and is proportional to any power of the coordinate in that direction. As the governing equation contains no term for the transverse force, the usefulness of the analytic results is very limited. *D. R. Bland (London).*

Sevlyakov, Yu. A. On uniqueness conditions for displacements of sloping spherical shells. Dopovidi Akad. Nauk Ukrain. RSR 1955, 448-450. (Ukrainian. Russian summary)

In his previous paper [same Dopovidi 1955, 235-237; MR 17, 212] the author solved the problem of a sloping spherical shell and obtained general formulas for displacements and internal forces containing six arbitrary constants for simply connected surfaces and seven arbitrary constants for multiply connected surfaces. When the surface of a shell is simply connected the six constants are determined from the boundary conditions. When the surface is multiply connected, like for instance a ring, the one additional constant in this case cannot be determined from the conditions on the outside and inside boundaries and the solution is not always unique. In this paper the author finds the necessary and sufficient conditions for a unique solution in the last case. *T. Leser.*

Weibel, Erich S. The strains and the energy in thin elastic shells of arbitrary shape for arbitrary deformation. Z. Angew. Math. Phys. 6 (1955), 153-189.

Part I of the paper is concerned with the analysis of strain in a deformable layer surrounding a reference surface in space. In the usual way it is assumed that displacements vary linearly with distance from the reference surface, that transverse shear and normal strains are negligible and that the thickness of the layer is small compared to a representative radius of curvature of the reference surface. On the basis of these assumptions the author obtains formulas for the changes in the coefficients of the first and second fundamental forms of the reference surface, and, in terms of these, expressions for direct and bending strains of the reference surface, taken along a certain set of mutually perpendicular directions which need not be tangent to the parametric curves on the reference surface. While formulas are general in the sense that no limitations are imposed on the choice of the parametric curves, they are limited by the omission of all non-linear terms. The author remarks that once expressions for strains and for strain energy have been obtained the remainder of the theory follows as a consequence of the energy principles. In Part II the general formulas of Part I are specialized so as to apply to (i) transverse bending of flat plates, using oblique cartesian coordinates, (ii) cylindrical shells, (iii) helicoidal shells. For this latter case a certain vibration problem is considered in detail. *E. Reissner (Cambridge, Mass.).*

★ Vekua, I. N. On the solution of the boundary problems of shell theory. Translated by Morris D. Friedman, 2 Pine St., West Concord, Mass., 1955. 5 pp. \$2.50. Translation of Soobšč. Akad. Nauk Gruzin. SSR 15 (1954), 3-6; MR 17, 104.

Tersenov, S. A. Asymptotic behavior of the eigenvalues and eigenfunctions of vibration of cylindrical shells. Soobšč. Akad. Nauk Gruzin. SSR 16 (1955), 11-18. (Russian)

Following the method of Carleman [Åttonde Skandinaviska Matematikerkongressen, Stockholm, 1934, Ohlsson, Lund, 1935, pp. 34-44] and using a method of estimating the compensating part of Green's function due to A. Pleijel [Ark. Mat. Astr. Fys. 27A (1940), no. 13; MR 2, 291], the author gets the asymptotic law

$$\lim n/\lambda_n = (8\pi(b+c))^{-1}(1+2b^{-1}(2b+c))J$$

for the eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ of a vibrating cylindrical shell of area J with elasticity constants b and c . The corresponding formulas for the eigenfunctions are also given. *L. Gårding (Lund).*

Nowiński, Jerzy. Thermal stresses in a thick-walled spherical vessel of transversally isotropic material. Arch. Mech. Stos 7 (1955), 363-374. (Polish. Russian and English summaries)

Melan, Ernst. Wärmespannungen bei der Abkühlung einer Kugel. Acta Phys. Austriaca 10 (1956), 81-86.

A uniform elastic sphere is initially at a uniform temperature. Suddenly, the surface temperature is decreased to a constant value. The paper solves the problem of the thermal stress distribution caused by heat conduction from the surface of the sphere. Results are exhibited graphically. *H. G. Hopkins (Sevenoaks).*

Pârvu, A. Une solution des déformations élastiques pour les corps à isotropie transverse. Rev. Univ. "C. I. Parhon" Politehn. București. Ser. Ști. Nat. 3 (1954), no. 4-5, 139-141. (Romanian. Russian and French summaries)

Rama, Silvio. Sul passaggio di un'onda elastica da uno ad un altro mezzo omogeneo ed isotropo. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 19(88) (1955), 507-526.

Two semi-infinite, homogeneous, isotropic media of different elastic properties are separated by a plane interface. The adhesion between the two media at this interface is not necessarily perfect. The paper gives a general discussion of the reflexion and transmission of plane waves incident on the interface. *H. G. Hopkins.*

Nakata, Yoshimoto; and Fujita, Hiroshi. On upper and lower bounds of the eigenvalues of a free plate. J. Phys. Soc. Japan 10 (1955), 823-824.

Let μ be Poisson's ratio. In a plane region D with a sufficiently regular contour C , consider the free plate problem (1) $\Delta \Delta w = \lambda_1^2 w$ in D , and

$$(2) \quad \begin{cases} \frac{\partial}{\partial n} [\Delta w + (1-\mu) \frac{\partial^2 w}{\partial s^2}] = 0 \text{ on } C; \\ \mu \Delta w + (1-\mu) \frac{\partial^2 w}{\partial n^2} = 0 \text{ on } C. \end{cases}$$

Let $L^2(D; \varrho)$ denote the Hilbert space of square-integrable functions over D with inner product $(u, v) = \iint_D u v \varrho dx dy$. Let $\mathfrak{H} = L^2(D; 1)$ and

$$\mathfrak{H}' = L^2(D; \mu) \times L^2(D; 1-\mu) \times L^2(D; 1-\mu) \times L^2(D; 2-2\mu).$$

Let $T(\mathfrak{H} \rightarrow \mathfrak{H}')$ be such that $Tu = (\Delta u, u_{xx}, u_{yy}, u_{xy})$. Let $\mathfrak{E} \subset \mathfrak{H}'$ be a set of 4-component vector functions (v_1, \dots, v_4) satisfying certain boundary conditions corresponding to (2). Define $T^*(\mathfrak{E} \rightarrow \mathfrak{H})$ such that

$$T^*v = \mu \Delta v_1 + (1-\mu)(v_{2xx} + v_{3yy} + 2v_{4xy}).$$

It is stated that the equation $T^*Tw = \lambda_1^2 w$ is equivalent to the system (1, 2).

A theorem of T. Kato [Math. Ann. 126 (1953), 253-262; MR 15, 326] is then applied to bound any λ_1 satisfying (1, 2). As a numerical example let D be the square with vertices $(\pm 1, \pm 1)$. The authors give bounds for the eigenvalue λ_0 corresponding to the vibration with nodal lines coincident with the two coordinate axes. With $\mu = 0.225$, and assuming that $18.5 < \lambda_1$, they claim that $3.418 < \lambda_0 < 3.554$. Details will be published later.

G. E. Forsythe (Los Angeles, Calif.).

Chakravorty, J. G. Torsional vibration of a cylinder of cylindrically aeolotropic material. Indian J. Theoret. Phys. 3 (1955), 17-20.

Frequency equations are formulated for torsional vibrations of a cylindrically aeolotropic cylinder which either is hollow or has an isotropic core. *Y.-Y. Yu.*

Vălcovici, V. Le calcul des tiges pesantes dans le forage des terrains pétrolifères. Rev. Univ. "C. I. Parhon" Politehn. București. Ser. Ști. Nat. 1 (1953), no. 2, 22-31. (Romanian. Russian and French summaries)

Prusov, I. A. Influence of a spring on the dynamical stress in a shaft hoisting cable. L'vov. Gos. Univ. Uč. Zap. 29, Ser. Meh.-Mat. no. 6 (1954), 98-104. (Russian)

Biot, M. A. General solutions of the equations of elasticity and consolidation for a porous material. J. Appl. Mech. 23 (1956), 91-96.

To explain the phenomenon of soil consolidation, K. Terzaghi [Erdbaumechanik auf bodenphysikalischer Grundlage, F. Deuticke, Leipzig-Wien, 1925] first proposed a simple one-dimensional model of a porous elastic material filled with fluid. This treatment was generalized by the author to the three-dimensional case, and general equations for both isotropic and anisotropic materials were established [J. Appl. Phys. 12 (1941), 155-164; 26 (1955), 182-185; MR 16, 643]. In the present paper general solutions are presented for the isotropic case. The solution for the displacement is obtained in terms of the Boussinesq-Papkovich functions which are used in the solution of the theory of elasticity and in terms of a function which satisfies the heat-conduction equation. The completeness of this solution of the consolidation theory is easily established, because that of the Boussinesq-Papkovich solution of the elasticity theory has been proved by R. D. Mindlin [Bull. Amer. Math. Soc. 42 (1936), 373-376]. A solution is also given for the two-dimensional case by means of a stress function, which is shown to be the sum of two functions satisfying respectively the biharmonic and the heat-conduction equations. General properties of the solutions are then examined, and the concept of consolidation mode is finally introduced from the viewpoint of eigenfunctions. [On the first page, $V_x V_y V_z$ should be $U_x U_y U_z$, and dilation should be dilatation.]

Y.-Y. Yu.

Grossman, P. U. A.; and Kingston, R. S. T. Mechanical conditioning of high polymers. Austral. J. Appl. Sci. 6 (1955), 442-452.

Mechanical conditioning refers to irreversible changes of properties due to application of mechanical stresses. The property here under consideration is the irrecoverable component of deformation ϕ_p . For mathematical convenience current theories put $\phi_p \sim Gt$ for constant stress G , although it is known that ϕ_p is a monotonic decreasing function of time. The present paper discusses a more realistic approach based on the "exhaustion theory" of metals, which allows one to consider the consequences of changes of stress in general cases where G is an arbitrary function of time. Various assumptions about the time-dependence of the irreversible effect are made and the corresponding creep and relaxation curves are calculated by the Laplace method. It is shown that comparison of the creep and relaxation functions cannot furnish information about the presence and time-dependence of mechanical conditioning.

B. Gross (Rio de Janeiro).

Sokolovskii, V. V. Some remarks on the plane problem of the theory of plasticity. Prikl. Mat. Meh. 18 (1954), 762-763. (Russian)

Ivlev, D. D. On the theory of simple deformation of plastic bodies. Prikl. Mat. Meh. 19 (1955), 734-735. (Russian)

The author points out that when the stress is a polynomial function of the total strain for small elastic-plastic deformations, proportional straining is not in general accompanied by proportional loading, and vice versa.

R. T. Shield (Providence, R.I.).

Panferov, V. M. Concentration of stresses in elastic-plastic deformations. *Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk* 1954, no. 4, 47-66 (1 plate). (Russian)

Novozhilov, V. V. On a class of compound loadings which are characterized by preservation of the directions of the principal axes. *Prikl. Mat. Meh.* 18 (1954), 415-424. (Russian)

Cristescu, N. Sur un problème de M. Lévy. *Acad. Repub. Pop. Romine. Bul. Şti. Sect. Şti. Mat. Fiz.* 7 (1955), 387-391. (Romanian. Russian and French summaries)

Eason, G.; and Shield, R. T. The influence of free ends on the load-carrying capacities of cylindrical shells. *J. Mech. Phys. Solids* 4 (1955), 17-27.

The paper is concerned with the axially symmetric plastic loading of thin circular cylindrical shells of finite length. The ends of the shell are assumed to be stress free. The types of loading considered are a ring of force and a band of uniform pressure. The basic equations used in the paper are due to Drucker [Proc. 1st Midwestern Conference Solid Mech., Univ. of Illinois, Urbana, Ill., 1953, pp. 158-163]. An approximate yield condition was used in the derivation of these equations. This approximation effects a considerable simplification in the present analysis without great loss in accuracy. *E. T. Onat.*

Eason, G.; and Shield, R. T. Dynamic loading of rigid-plastic cylindrical shells. *J. Mech. Phys. Solids* 4 (1956), 53-71.

The dynamic loading of circular cylindrical shells has been previously discussed by the reviewer [e.g., same *J.* 3 (1955), 176-188; MR 17, 214] for shells which are uniformly loaded over their length, but with arbitrary time variation of load. Here the authors consider a ring of force and a finite band of pressure. The time variation

of load is discussed generally, and particular applications are made for rectangular and triangular pulses. In addition, a ring load or band of constant magnitude, moving with constant velocity along an infinite tube is considered. Results are expressed in terms of simple functions and descriptive curves. *P. G. Hodge, Jr.*

Shield, R. T. On Coulomb's law of failure in soils. *J. Mech. Phys. Solids* 4 (1955), 10-16.

The Coulomb law of failure for an ideal cohesive soil is interpreted by the author to obtain the yield surface for three-dimensional stress fields. The author's interpretation is similar to that of Guest [Phil. Mag. (7) 30 (1940), 349-369]. However the author is careful to note that the yield surface is a pyramid which has irregular hexagons for its sections normal to its axis. This axis is equally inclined to the $\sigma_1, \sigma_2, \sigma_3$ axes where σ_1, σ_2 and σ_3 are principal stresses. The author then makes the assumption of perfect plasticity in order to derive the associated flow rule. This assumption constitutes an extension of the work of Drucker and Prager on two-dimensional problems of soil mechanics [Quart. Appl. Math. 10 (1952), 157-165; MR 13, 1007].

The last part of the paper is concerned with the application of the Coulomb yield criterion to obtain a lower bound for the bearing capacity of a rectangular footing on the plane surface of a semi-infinite mass of soil. The bound is obtained by application of a theorem of limit analysis [Drucker, Prager and Greenberg, *ibid.* 9 (1952), 381-389; MR 13, 603]. The stress field used in the application of this theorem is an adaptation of the statically admissible two-dimensional stress field previously used by Shield and Drucker [*J. Appl. Mech.* 20 (1953), 453-460].

Finally the author notes that this bound applies to any footing which has a convex area of contact. *E. T. Onat.*

See also: Litvinov, p. 792; Rivlin, p. 797.

MATHEMATICAL PHYSICS

Vallée, Robert. Une point de vue algébrique en théorie macroscopique de l'observation. *C. R. Acad. Sci. Paris* 241 (1955), 179-180.

„On considère des opérateurs \mathfrak{A} et \mathfrak{B} qui réduisent dans une observation, respectivement, le „champ” spatio-temporel et l'ensemble des „fréquences” (spatiales et temporelles). Au produit et à la somme de deux opérateurs \mathfrak{A} (ou \mathfrak{B}) correspondent, respectivement, la „mise en série” et la „mise en parallèle” des deux dispositifs d'observation associés.”

This paper is concerned with a generalized interpretation of Fourier transforms such as used in the determination of crystal structure. *C. C. Torrance.*

Chakrabarty, S. K.; and Gupta, M. R. Calculations on the cascade theory of showers. *Phys. Rev. (2)* 101 (1956), 813-819.

Let $P(E, t)dE$ be the mean number of electrons and positrons in a cascade shower at depth t in the energy range $(E, E+dE)$, and let $Q(E, t)dE$ be the corresponding quantity for photons. Then P and Q satisfy the well-known integro-differential equations of cascade theory. Bhabha and Chakrabarty obtained a solution in the form of an infinite series [Phys. Rev. (2) 74 (1948), 1352-1363]. By rearranging the earlier series, a series superior to the original one for computation is obtained. Each term has the form of a complex integral which, for computational

purposes, is evaluated by the saddle-point method. The first term gives almost the entire contribution of the series. Numerical calculations are given for P and for $N(E, t) = \int_0^t P(x, t)dx$. Analytic approximations for large and small E are given. *T. E. Harris.*

See also: Deprit, p. 705; Sneddon, p. 732.

Optics, Electromagnetic Theory, Circuits

★ **Glaser, Walter.** Grundlagen der Elektronenoptik. Springer-Verlag, Wien, 1952. x+699 pp. DM 120.00, \$ 28.60.

In the last decade many excellent books have been written on electron optics. However, with the exception of de Broglie's book "Optique électronique et corpusculaire" [Hermann, Paris, 1950] and Sturrock's monograph "Static and dynamic electron optics" [Cambridge, 1955; MR 16, 1180] the emphasis has been more on the experimental and less on the analytical foundations of the subject. Glaser's book is unique in the sense that it is the most comprehensive treatise ever written on the mathematical theory of electron optics. The book is divided into three parts. The first part dealing with "image properties of electromagnetic fields and Gaussian dioptrics" extends over half of the book. Among the topics treated are motion of charged particles in a general electromagnetic field,

derivation of the paraxial ray equation (p.r.e.), optical imagery of rotational symmetric electrostatic and magnetostatic fields, a comprehensive analytical treatment of potential field distributions and methods used in field plotting and ray tracing. The treatment of electrostatic and magnetic lenses and their image properties based on the p.r.e. is thorough. Among other topics included are numerical integration of the p.r.e., chromatic aberration in electron lenses and electron lenses satisfying Newton's image equation. Of special importance is the mathematical analysis of electric and magnetic field (lens) models for which rigorous solutions can be obtained.

In the second part the "theory of geometrical electron aberrations" is developed from the Hamiltonian method. The analysis is limited to third-order aberrations, these being the simplest kind which undoubtedly play the most important role in the formation of the image. Starting with the analogy between ray optics and mechanics based on the variational principles of Fermat and Hamilton, the author derives an expression of the refractive index and characteristic functions associated with electron-optical systems. For rotational symmetric fields, the aberration coefficients are obtained from Seidel's eiconal function. The author then gives a clear analysis of individual aberration curves formed on the Gaussian image plane and their efforts in the quality of the image. Explicit expressions of the aberration coefficients are exhibited for a bell-shaped magnetic lens. An elegant treatment of caustic surfaces in electron lenses is also included, a topic hardly mentioned in books on this subject. This is followed by an analysis of intensity distribution curves (isophotes-characteristic curves) in the image space. Separate chapters are devoted to the aberrations produced by distorted rotational symmetric fields and deflecting systems. Here the author makes use of the method of variation of constants and the mixed characteristic function in obtaining the aberrations. A brief treatment of the focusing of a circular electron bundle is also included.

The last part, comprising a fourth of the book, is devoted to the "geometrical theory of diffraction", largely due to the efforts of the author and his collaborators. After a brief introduction into the corpuscular and wave behavior of electrons, the author derives Schrödinger's equation of the electron from the Hamilton-Jacobi theory, expressions for charge and current density and various forms of Heisenberg's uncertainty relations. The Schrödinger equation for a free-field particle is then applied to the problem of diffraction by a slit. After a brief but clear presentation of the Huyghens-Kirchhoff principle to diffraction by various apertures, the author develops the theory of electron diffraction based on the paraxial Schrödinger equation, including applications to several types of apertures and screens [Glaser and Schiske, *Ann. Physik* (6) 12 (1953), 240-266; MR 15, 375]. Then follows a chapter on geometrical electron optics as an approximation to wave mechanics. Of special interest is the theory of electron-optical images based on wave mechanics which is developed along similar lines given by Debye and Picht for wave optics, with applications to electron microscopy [see Glaser, Österreich. Akad. Wiss. Math.-Nat. Kl. S-B. IIa. 159 (1950), 297-360; MR 14, 931]. In the final chapter, the author formulates the diffraction theory of aberrations of electron optics by extending Kirchhoff's formulation to Schrödinger's equation of the electrons in an electromagnetic field [Glaser and Schiske, *Ann. Physik* 12 (1953), 267-280; MR 15, 375].

The text contains many excellent illustrations, author and subject indices and an extensive bibliography with brief commentaries by the author. Scientists working in the field of dynamic electron optics will be somewhat disappointed in Glaser's treatise since it contains not even a single chapter on the electron optics of non-stationary electromagnetic fields. One may hope that the author will eliminate this deficiency in a companion volume in the not too distant future. In the opinion of the reviewer this book fulfills a double purpose. From the didactic side it provides the student and the non-specialist with more than the fundamentals of electron optics and the specialist with a reference work of enduring value in theoretical foundations of the subject. N. Chako (New York, N.Y.).

★ Klemperer, O. *Electron optics*. 2d ed. Cambridge, at the University Press, 1953. xiii+471 pp. \$9.50.

In this book Klemperer presents electron optics from the experimental and the practical rather than the analytical side. For this reason it should appeal to a larger circle of readers than Glaser's treatise reviewed above. The author has presented the essentials of the theory of electron optics in a clear manner and in addition has included extensive new material not easily accessible in books. In this respect it should be valuable to physicists and engineers working in this field; it should also be useful to students with limited mathematical background who would be interested in learning the principles and applications of electron optics. The book may be divided into three main parts: Principles of electron optics and electron lenses; aberrations; electron-optical instruments and applications. The first part consisting of five chapters includes, after a historical introduction, such topics as cardinal and focal points and planes in electron-optical systems, numerical and practical methods of plotting field distributions, ray tracing, derivation of the paraxial ray equation (p.r.e.), the focal-length formula for thin electron lenses and an adequate and clear analysis and discussion of electron-optical properties of various types of rotational symmetric electrostatic and magnetostatic lenses, based on the paraxial ray equation.

The second part (Chapters 6-8) contain a descriptive account of third-order geometrical electron aberrations, a detailed analysis of chromatic aberration and space-charge effects on the formation of images. In the last four chapters (third part) the author gives a full description of various types of electron sources, the design of emission systems and a clear analysis of two-dimensional fields producing a line focus. Deflecting systems with homogeneous and non-homogeneous field distributions are fully treated in view of their important role in the design of cathode-ray oscillographs, photomultipliers, mass spectrometers and other instruments which are extensively used in research laboratories and in industry. In the concluding chapter the author presents a general survey of the various applications of electron optics in different fields of research and industrial establishments.

The author has included an extensive up-to-date bibliography which increases its usefulness as a reference work. N. Chako (New York, N.Y.).

Herzberger, M. *Intrinsic image-error theory*. J. Opt. Soc. Amer. 46 (1956), 132-138.

This paper is concerned with the connection between aberrations defined by means of a characteristic function and data that are obtained from ray trace. Third- and fifth-order aberrations are considered. E. Wolf.

Stettler, R. Über die optische Abbildung von Flächen und Räumen. *Optik* 12 (1955), 529-543.

The author proves that media with central symmetry can be found, in which a given concentric sphere is imaged onto another arbitrarily given concentric sphere. Moreover, he shows that a class of systems can be found which give a sharp image of all the points of space, a class which includes the fish-eye of Maxwell. A paper by Boegehold and Herzberger [*Z. Angew. Math. Mech.* 15 (1935), 157-178] should be consulted. *M. Herzberger.*

De, M. The influence of astigmatism on the response function of an optical system. *Proc. Roy. Soc. London. Ser. A.* 233 (1955), 91-104.

The author studies the response function as a function of the line frequency for various values of an astigmatic image, giving tolerances for astigmatism and curvature of field. He compares the results of the study including diffraction with one based on geometrical optics alone, and claims that the agreement holds good only for low frequencies. *M. Herzberger* (Rochester, N.Y.).

Linfoot, E. H. La "lumière diffractée éloignée" et l'appréciation des images. Avec un Note de P. Michel Duffieux. *Rev. Opt.* 34 (1955), 617-631.

The author discusses ideas, proposed by Duffieux and Lansraux, to use the radius of gyration as a measure of the image quality in an optical image. The result of his investigation is that different methods have to be applied to judge the influence of the nucleus and of the halo of the geometric optical, and the diffracted image, and that the center of inertia of both do coincide only in special cases, as the author shows by mathematical analysis.

An attached note of Duffieux's draws attention to later work of his, which is basically in agreement with Linfoot's analysis. *M. Herzberger* (Rochester, N.Y.).

Giovanelli, R. G. Reflection by semi-infinite diffusers. *Opt. Acta* 2 (1955), 153-162.

Discussion of total and directional reflectance of semi-infinite diffusers of constant composition, scattering according to a phase function of the form $\omega_0(1+x \cos \theta)$. Diffusers whose matrices have unit refractive index as well as diffusers whose matrices exceed unity are considered. Results are given in tabulated form, with accuracy claimed to be of order of ± 0.001 or better. *E. Wolf.*

Teisseyre, R. General solutions for the diffraction of a dipole field by a perfectly conducting wedge. *Bull. Acad. Polon. Sci. Cl. III.* 3 (1955), 523-526.

The author derives the electromagnetic field for the case of an electric or magnetic dipole in the presence of a conducting wedge. *A. E. Heins* (Pittsburgh, Pa.).

Filipovič, V. N.; and Poral-Košic, E. A. On the theory of scattering of Roentgen rays by macroscopic isotropic bodies. *Dokl. Akad. Nauk SSSR (N.S.)* 105 (1955), 968-971. (Russian)

Vacca, Maria Teresa. Sulla propagazione di onde elettromagnetiche in un tubo cilindrico circolare riempito di dielettrico eterogeneo. *Univ. e Politec. Torino. Rend. Sem. Mat.* 14 (1954-55), 297-310.

The problem discussed in this paper is that of the propagation of electromagnetic waves inside a perfectly conducting straight wave-guide of circular cross-section,

filled with a heterogeneous dielectric. The permeability of the dielectric is constant, but the dielectric constant is of the form $\epsilon = \epsilon_0 + \epsilon_1(r)$, where ϵ_0 is a constant, but ϵ_1 is a function of the distance r from the axis of the guide. Since ϵ_1 and its first derivative are supposed small, the problem is a perturbation problem. *E. T. Copson.*

★Zernov, N. V. On the radiation of electromagnetic waves from a circular waveguide with an infinite flange. [On the question of the diffraction of plane-cylindrical electromagnetic waves.] Translated by Morris D. Friedman, 2 Pine St., West Concord, Mass., 1955. 14 pp. Translation of *Ž. Tehn. Fiz.* 21 (1951), 1066-1075; *MR* 14, 702.

Wait, James R. Reflection at arbitrary incidence from a parallel wire grid. *Appl. Sci. Res. B.* 4 (1955), 393-400.

The author has derived a general solution for the problem of an arbitrarily polarized plane electromagnetic wave incident obliquely on a planar grid. The solution when specialized agrees with earlier results for perpendicular incidence, and polarization parallel or perpendicular to the grid. The method is essentially classical and appeals to the periodicity of the secondary field scattered by the grid. A transformation of the resulting series of Hankel functions brings it into a tractable and more rapidly converging form. *W. K. Saunders* (Washington, D.C.).

Fedorov, F. I. On superposition of waves with different polarizations. *Izv. Akad. Nauk Belorussk. SSR* 1955, 109-118. (Russian)

In an antecedent paper [*Izv. Akad. Nauk Belorussk. SSR* 1954, no. 6, p. 83ff.] the author has formulated in invariant vector form criteria characterizing the state of polarization of plane electromagnetic waves; for example, linear polarization of the electric vector \mathbf{E} is characterized by the equation $\mathbf{E}_1 \times \mathbf{E}_2 = 0$, where $\mathbf{E}_1, \mathbf{E}_2$ are respectively the real and imaginary parts of the amplitude of \mathbf{E} . These relations are here used to discuss the polarization of the wave obtained by superposing two waves of different polarizations. A typical result is a proof that it is always possible to resolve a wave of arbitrary polarization unambiguously into two circularly polarized waves with opposite rotations. The methods and results are all quite elementary. *R. N. Goss* (San Diego, Calif.).

Nomura, Yūkichi; and Takaku, Kōshun. On the propagation of the electromagnetic waves in an inhomogeneous atmosphere. *J. Phys. Soc. Japan* 10 (1955), 700-714.

The atmosphere is assumed to be composed of concentric spherical layers, in each of which the dielectric constant ϵ is proportional to r^{2m} , where r is the distance from the center of the earth and m is a constant depending on the layer. The field produced by an electric or magnetic dipole in such a medium is obtained by solving Maxwell's equations. (A similar problem solved by the reviewer [Comm. Pure Appl. Math. 4 (1951), 317-350; *MR* 13, 408] seems to have been overlooked.) Using the methods of van der Pol and Bremmer [*Phil. Mag.* (7) 24 (1937), 825-864], the authors express the field as a contour integral and evaluate it by the method of saddle points. Thus they obtain the well-known result that in the "lit" region the field can be usefully approximated by the geometrical-optical methods of ray tracing.

B. Friedman (Berkeley, Calif.).

Tonolo, Angelo. Sull'integrazione delle equazioni di propagazione delle onde elettromagnetiche nei mezzi omogenei isotropi e cristallini. *Ann. Mat. Pura Appl.* (4) 39 (1955), 39-61.

This paper is concerned with the solution of the Cauchy problem for the equations governing the propagation of electromagnetic waves, that is, with finding the electric and magnetic vectors at some instant $t(>0)$ given the values everywhere at the initial instant $t=0$. Three cases are considered: (i) a homogeneous isotropic conducting medium; (ii) a homogeneous uniaxial crystalline non-conducting medium; (iii) a homogeneous biaxial crystalline non-conducting medium.

The tools used are: (i) Poisson's solution of the initial-value problem for the wave equation; (ii) Weber's solution of the initial-value problem for

$$\Delta_2 U - \frac{\partial^2 U}{\partial t^2} + \alpha U = 0;$$

(iii) the solution of Grünwald and Signorini of the initial-value problem for

$$\begin{aligned} \frac{\partial^2 X}{\partial t^2} - a^2 \Delta_2 X - (a^2 - c^2) \frac{\partial}{\partial y} \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} \right) &= 0, \\ \frac{\partial^2 Y}{\partial t^2} - a^2 \Delta_2 Y + (a^2 - c^2) \frac{\partial}{\partial x} \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} \right) &= 0; \end{aligned}$$

(iv) the formulae of Herglotz for the solution of the initial-value problem for the equation of wave motions in a crystalline medium. *E. T. Copson* (St. Andrews).

Mihalevskii, V. S.; and Gorelov, M. I. On the radiation of an electron moving along the axis of a coaxial spiral curve. *Rostov. Gos. Univ. Uč. Zap. Fiz.-Mat. Fak.* 32 (1955), no. 4, 157-160. (Russian)

Agostinelli, Cataldo. Onde elettromagnetiche guidate entro un tubo cilindrico percorso da un fluido dielettrico in moto traslatorio uniforme. *Univ. e Politec. Torino. Rend. Sem. Mat.* 14 (1954-55), 257-268.

The equations of the electromagnetic field in a dielectric moving with uniform velocity \mathbf{V} referred to an inertial system are

$$\mathbf{B} = -c \operatorname{curl} \mathbf{E}, \quad \mathbf{D} = c \operatorname{curl} \mathbf{H}, \quad \operatorname{div} \mathbf{B} = 0, \quad \operatorname{div} \mathbf{D} = 0,$$

where

$$\mathbf{B} = \left[\mu(1 - \beta^2) \mathbf{H} - \frac{n^2 - 1}{c} \left\{ \frac{\mu}{c} (\mathbf{H} \cdot \mathbf{V}) \mathbf{V} + \mathbf{V} \times \mathbf{E} \right\} \right] / (1 - n^2 \beta^2),$$

$$\mathbf{D} = \left[\epsilon(1 - \beta^2) \mathbf{E} - \frac{n^2 - 1}{c} \left\{ \frac{\epsilon}{c} (\mathbf{E} \cdot \mathbf{V}) \mathbf{V} - \mathbf{V} \times \mathbf{E} \right\} \right] / (1 - n^2 \beta^2),$$

in the usual notation (which differs slightly from that used by the author); here ϵ is the dielectric constant, μ the permeability, $n^2 = \epsilon\mu$, $\beta = V/c$.

If \mathbf{V} is in the direction of the axis of z in a rectangular coordinate system, it follows that

$$\begin{aligned} B_x &= \mu^* H_x + \beta^* E_y, & D_x &= \epsilon^* E_x - \beta^* H_y, \\ B_y &= \mu^* H_y - \beta^* E_x, & D_y &= \epsilon^* E_y + \beta^* H_x, \\ B_z &= \mu H_z, & D_z &= \epsilon E_z, \end{aligned}$$

where

$$\epsilon^* = \epsilon \frac{1 - \beta^2}{1 - n^2 \beta^2}, \quad \mu^* = \mu \frac{1 - \beta^2}{1 - n^2 \beta^2}, \quad \beta^* = \beta \frac{n^2 - 1}{1 - n^2 \beta^2}.$$

If such a dielectric is a homogeneous isotropic fluid moving uniformly inside a perfectly conducting fixed

straight wave guide, the conditions on the walls of the guide are

$$E_z = 0, \quad \mathbf{E} \cdot \mathbf{T} - \beta \mathbf{B} \cdot \mathbf{N} = 0,$$

where the generators of the guide are parallel to O_z , \mathbf{N} is the unitvector normal to the guide, \mathbf{T} the unit vector tangential to the cross-section of the guide. When monochromatic waves are propagated along the guide in this moving fluid, the field components involve z and t only in a factor $\exp(i\omega t - i\alpha z)$ where α is a real constant. Quafuncions of x and y , E_x and H_x satisfy

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \lambda^2 \phi = 0,$$

where

$$\lambda^2 = \frac{\omega^2 \epsilon^* \mu^* - (\omega \beta^* + c\alpha)^2}{c^2} \cdot \frac{1 - n^2 \beta^2}{1 - \beta^2},$$

and all the other components are expressible in terms of derivatives of E_x and H_x . The boundary conditions on the walls of the guide reduce to

$$E_z = 0, \quad \partial H_x / \partial N = 0,$$

just as in the case $V=0$.

E. T. Copson.

Ilkovič, Dionýz. Contribution to the formulation of the basic laws of electrodynamics in Minkowskian four-dimensional space-time. *Mat.-Fyz. Časopis. Slovensk. Akad. Vied* 5 (1955), 222-227. (Slovak)

En partant des équations de Maxwell on déduit les formules fondamentales bien connues du champ électromagnétique dans l'espace de Minkowski à quatre dimensions. L'auteur montre une méthode qui permet d'utiliser le calcul vectoriel et tensoriel direct pour déduire les lois fondamentales. *F. Vyěichlo* (Prague).

Mayer, Daniel. Remark on the choice of a space in the theory of the electromagnetic field. *Mat.-Fyz. Časopis. Slovensk. Akad. Vied* 5 (1955), 228-230. (Slovak)

Dans cette remarque on montre qu'on peut formuler dans l'espace de Minkowski à quatre dimensions les lois fondamentales électromagnétiques plus généralement que dans l'espace ordinaire à trois dimensions. *F. Vyěichlo*.

Gavrila, M. Une démonstration des formules de Liénard-Wiechert. *Rev. Univ. "C. I. Parhon" Politehn. București. Ser. Ști. Nat.* 1 (1953), no. 2, 57-61. (Romanian. Russian and French summaries)

Ashour, A. A. Note on the problem of the electrified disc. *Proc. Edinburgh Math. Soc.* (2) 10 (1956), 123-124.

The problem of a circular disc maintained at potential V_0 is solved by oblate spheroidal coordinates.

A. E. Heins (Pittsburgh, Pa.).

★ Grindberg, G. A.; and Bonshtedt, B. E. Principles of the exact theory of the wave field of a transmission line. Translated by Morris D. Friedman, 2 Pine St., West Concord, Mass., 1955. 40 pp.

Translation of *Ž. Tehn. Fiz.* 24 (1954), 67-95; MR 16, 98.

De Schwarz, M. J. Tensioni e correnti in una linea pupinizzata dissipativa. *Ann. Mat. Pura Appl.* (4) 40 (1955), 349-364.

Determination of the voltage and current in an electric

circuit consisting of N equal parts, where between each two parts and also at both ends are inserted Pupin bobbins. The problem is solved by means of Laplace transformation. Special cases have been treated by L. Amerio [Pont. Acad. Sci. Comment. 4 (1940), 83-145; MR 2, 334] and by M. J. De Schwarz and M. L. Ventura Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 89 (1955), 193-207; MR 17, 110]. Now the author deduces, using Riemann's formula for the inverse Laplace-transform

$$V_{N,k}(t) = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} e^{pt} F_{N,k}(p) dp \quad (k=1, 2, \dots, N-1),$$

$$I_{N,k}(t) = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} e^{pt} G_{N,k}(p) dp \quad (k=0, 1, \dots, N),$$

where $F_{N,k}$ and $G_{N,k}$ are functions determined by Amerio (loc.cit.), $V_{N,k}$ is the voltage, $I_{N,k}$ the current in the centre of the k th bobbin. The first of these integrals is divergent for $k=1$, the second for $k=0$, in these cases the principal values in the sense of Cauchy are to be taken. Further, the limiting case $N \rightarrow \infty$ is considered.

For the finite circuit the voltages and currents are also expressed in the form of series. *H. Bremekamp.*

Cartianu, Gh. La résolution des équations du circuit oscillant à paramètres variables avec la temps, avec application à la modulation de fréquence. Rev. Univ. "C. I. Parhon" Politehn. București. Ser. Ști. Nat. 4 (1955), no. 8, 107-114. (Romanian. Russian and French summaries)

de Buhr, Johann. Die geometrische Darstellungsweise kombinierter, linearer Vierpole. Arch. Elek. Übertr. 9 (1955), 561-570.

A correspondence is established between the impedance transform due to a linear four-terminal network and the equations of motion of a rigid body in non-Euclidean kinematics. The representation for cascade connections of several such networks can therefore be obtained from geometrical considerations. Applications to pure reactance systems show the usefulness of the method.

B. Gross (Rio de Janeiro).

Bauer, Friedrich L. Die Betriebs-Kettenmatrix von Vierpolen. Arch. Elek. Übertr. 9 (1955), 559-560.

The cascade matrix of a four-terminal network provides for a simple description of the cascade connection of several networks. But the description of the operational characteristics of a complex network is given in terms of the transfer function and other characteristic functions and not in terms of the parameters of the cascade matrix. It is shown that these operational quantities constitute the elements of a single matrix that is obtained from the cascade matrix by affine transformation and is called the operation cascade matrix. *B. Gross (Rio de Janeiro).*

Mogensen, A. A general theory of reactive non-dissipative L -sections. Kungl. Tekn. Högsk. Handl. Stockholm no. 95 (1955), 60 pp.

Pure reactance ladder networks built of L -type filter sections are discussed. Attenuation, phase constant characteristics, matching possibilities, and design formulae are studied. The theory is based on Foster's reactance theorem; a single frequency transform gives the characteristic functions for all theoretically possible reactance sections. Chapter I summarizes Foster's theorem and its implications. It discusses frequency substitutions in which

ω is replaced by a function $H(\omega)$. Chapter II describes different types of reactance filters and their properties. Chapter III deals with frequency transformations defined as an operation in which all impedances of the original network are multiplied by the same dimensionless function; each frequency is then replaced and multiplied. Chapter IV applies the previous results to a discussion of L -sections and their attenuation characteristics. Several examples are given. *B. Gross (Rio de Janeiro).*

Brazma, N. A. Generalization of theorems of variation and compensation for n parameters of an electric circuit. Dokl. Akad. Nauk SSSR (N.S.) 105 (1955), 271-274. (Russian)

A linear network is studied n of whose branch impedances Z_i ($i=1, \dots, n$) undergo a variation δZ_i . The ensuing variations δI_i of the branch currents I_i are found. The effect of the variations δZ_i is equivalent to introducing voltage generators with EMF's of $(I_i + \delta I_i) \delta Z_i$ into the n branches [compensation theorem]. The unperturbed network is characterized in terms of the matrix Y consisting of the short-circuit transadmittances among the equivalent generators. Then, forming the diagonal matrix δZ of the δZ_i 's, and the column matrices I and δI of the I_i 's and δI_i 's respectively, the equation is derived:

$$\delta I = (E + Y \delta Z)^{-1} I - I,$$

where E is the unit matrix. An approximate analysis for small δZ_i 's is indicated by means of a matrix expansion of the above equation. *H. A. Haus (Cambridge, Mass.).*

Hohn, Franz. Some mathematical aspects of switching. Amer. Math. Monthly 62 (1955), 75-90.

Exposé élémentaire sur les applications de l'algèbre de Boole à la théorie des réseaux d'interrupteurs. Bibliographie. *J. Riguet (Paris).*

See also: Kline, p. 747; Obukhov, p. 798; MacLean, p. 799.

Quantum Mechanics

Ekstein, H. Theory of time-dependent scattering for multichannel processes. Phys. Rev. (2) 101 (1956), 880-890.

In this paper the usual time-dependent scattering theory is extended to multichannel processes with a view towards applications in relativistic field theory. Thus, when compared to the well-known work of Wigner and Eisenbud [Phys. Rev. (2) 72 (1947), 29-41] and Wigner [ibid. 73 (1948), 1002-1009] the importance of configuration space is suppressed and the internal and external region is not sharply separated by a surface which offers the characteristic boundary-value problem. Instead, the starting point is the Schrödinger equation. The Schrödinger wave functions are expressed in terms of the basic functions Φ_n which characterize the channels, and they satisfy asymptotic conditions. The Φ_n are not orthogonal and are not linearly independent in general. This leads to the result that, although an S -matrix can be defined as usual, an S -operator cannot be defined as in the usual single-channel case. Though the author claims that no linear S -operator exists, he proves this statement only with respect to a certain definition of this operator. This reviewer has seen a later (unpublished) paper by the author in which an S -operator for the multi-channel case

is indeed defined (following the Yang-Feldman approach). The advantages of the time-dependent formalism over the time-independent one are pointed out. Wave packets are used throughout the theory. It is shown that the usual one-channel theory is contained in this formulation as a special case. [See, e.g., M. Gell-Mann and M. L. Goldberger, *Phys. Rev.* (2) 91 (1953), 398-408; MR 15, 382.]

F. Rohrlich (Iowa City, Ia.).

Taylor, J. C. The form of the divergencies in quantum electrodynamics. *Proc. Roy. Soc. London. Ser. A.* 234 (1956), 296-300.

A new derivation of the results of Landau, Abrikosov, Halatnikov [Dokl. Akad. Nauk SSSR (N.S.) 95 (1954), 497-500, 773-776, 1177-1180; MR 16, 315, 316] is given with the aid of the functional equations of Gell-Mann and Low [Phys. Rev. (2) 95 (1954), 1300-1312; MR 16, 315] for the asymptotic forms of the Green's functions. It seems to be the author's opinion that the rigour of the argument is improved on in this way.

G. Källén.

Rzewuski, J. On differential structure of non-local field theories. *Bull. Acad. Polon. Sci. Cl. III.* 2 (1954), 429-433 (1955).

The equation for a charged scalar field in non-local interaction with an external field is shown, by expanding in a sort of Neumann series, to be equivalent to a certain differential equation and hence some general information about existence and uniqueness is obtained. The method is illustrated also on the problem of charged and neutral scalar fields interacting non-locally.

A. J. Coleman.

Arnoux, E.; and Heitler, W. The self-stress problem and the limits of validity of quantized field theories. *Nuovo Cimento* (10) 2 (1955), 1282-1296.

This paper consists of two parts. In the first part the authors generalize previous work by others on the self-stress to the case of interacting photons, electrons, scalar and pseudoscalar meson, and nucleons, including both types of meson-nucleon interaction as well as all five β -decay interactions. In all cases one obtains a vanishing self-stress for each elementary particle provided some invariant cut-off method is used consistently. This follows from covariance and dimensional arguments, but can also be shown explicitly to all orders of perturbation theory [Y. Takahashi and H. Umezawa, *Progr. Theoret. Phys.* 8 (1952) 193-204; MR 14, 709].

In the second part it is argued that a comparison of the second-order electromagnetic self-energy of spin-zero mesons with the experimental mass difference of charged and neutral π -mesons is significant. This comparison leads to a cut-off momentum for the integration over intermediate momenta of about $\frac{1}{2}Mc$ where M is the proton mass. It is then shown that a breakdown of quantum electrodynamics at such momenta in the center of momentum system leads to effects which are far below the present experimental accuracy. Finally, the failure of present meson theory to yield satisfactory quantitative results is attributed to a break-down near $\frac{1}{2}Mc$ which does not leave enough range of validity above the meson rest energy of about $\frac{1}{2}Mc^2$: No significant range of integration remains.

F. Rohrlich (Iowa City, Ia.).

Watanabe, Ichie. On the renormalization of Heisenberg treatment. *Progr. Theoret. Phys.* 14 (1955), 151-165.

By using the Heisenberg representation, many authors have been able to obtain a set of coupled integral equations

which lead to non-perturbation approximations for field-theoretical problems. In this paper the problem of subtracting the divergent parts from those equations is dealt with. The renormalization prescription is given in such a way that, if a perturbation expansion is performed, the result reduces to the Dyson renormalized power series.

S. Fubini (Turin).

De Witt, Bryce S. State-vector normalization in formal scattering theory. *Phys. Rev.* (2) 100 (1955), 905-911.

The author studies the familiar problem of state vector normalization using the Lippman-Schwinger time-independent scattering theory. As is well-known in field theory, this problem is closely connected with charge renormalization. The results of this paper are quite interesting for two reasons: first, because the assumptions under which the scattering theory is valid for field-theoretical problems are formulated in a general way; secondly because the rules for the construction of a renormalized S matrix do not depend on any perturbation assumption.

S. Fubini (Turin).

Johnson, Kenneth A. Renormalization of the mass operator. *Phys. Rev.* (2) 101 (1956), 448-451.

The proof that the particle Green's function can be expressed in renormalized variables, depends on establishing the presence of the extra Z factor in the self-energy parts, which arises from the b -divergencies. This is accomplished by formal manipulations with functional derivatives, without the use of perturbation theory. The question of the finiteness, or otherwise, of the renormalized expression outside the framework of perturbation theory, is not discussed.

P. T. Matthews (Birmingham).

Taylor, J. C. Tamm-Dancoff method. *Phys. Rev.* (2) 95 (1954), 1313-1317.

Working with a covariant formulation, it is shown that the new Tamm-Dancoff approximation proposed by Dyson [Phys. Rev. (2) 91 (1953), 1543-1550; MR 15, 768] is not renormalizable. A new renormalizable equation is proposed, which is related approximately to the new Tamm-Dancoff equation, as the Levy-Klein approximation was related to the old Tamm-Dancoff equation.

P. T. Matthews (Birmingham).

Klein, Abraham. New Tamm-Dancoff formalism. *Phys. Rev.* (2) 95 (1954), 1676-1682.

The one-particle problem is treated by the new Tamm-Dancoff method, and rules are given for constructing the kernel of the integral equation for the leading amplitude, to any order in the coupling constant. This equation involves singularities at the thresholds for unreal processes. These spurious singularities can be removed, to fourth order, by partial fraction decomposition, but the equation so transformed is not equivalent to the covariant equation to the same order.

P. T. Matthews.

Dirac, P. A. M. The stress tensor in field dynamics. *Nuovo Cimento* (10) 1 (1955), 16-36.

The stress tensor $T_{\mu\nu}$ is such that if n^μ is the normal to a space-like surface, then $T_{\mu\nu}n^\mu n^\nu$ is the energy-density in the surface. Dirac shows that this may be identified with the Hamiltonian density function only when the latter is independent of the curvature and parametrization of the surface. A necessary condition for this is that $[T_{\mu\nu}(P), T_{\mu\nu}(Q)]$ involve the delta function and its derivatives to at most the first order.

A. J. Coleman (Toronto, Ont.).

Schulz, William Donald. Interaction of nonlocal and local fields. Phys. Rev. (2) 99 (1955), 290-301.

The paper considers the interaction of a scalar non-local field of the type proposed by Yukawa [Phys. Rev. (2) 91 (1953), 415-416; MR 15, 382] with an ordinary local spinor field. The non-local free field equation is

$$F\phi(X, r) = 0; F = \frac{-\partial^2}{\partial X_\mu \partial X_\mu} + \frac{\lambda}{2} \left(\frac{-\partial^2}{\partial r_\mu \partial r_\mu} + \frac{r_\mu r_\mu}{\lambda^4} \right), \\ X_\mu = \frac{1}{2}(x_\mu' + x_\mu''); r_\mu = (x_\mu' - x_\mu'').$$

Here x_μ', x_μ'' ($\mu=1, 2, 3, 4$) stand for two sets of space-time parameters. In order to eliminate unphysical "mass states" defined by the operator F , it is necessary to define an invariant "projection operator" P such that $P\phi(X, r)$ are the only physical state interacting with the spinor field. The interaction between the fields is so chosen that the system is restricted to involve only mesons of spin zero, in which case it is shown to be equivalent to a system of local fields coupled through a non-local interaction. The problem of convergence of the proposed theory is therefore equivalent to the corresponding problem in the theory of non-local interaction of local field, and all the difficulties encountered there remain here.

Nothing is said, however, about the difficulties concerning the causality of the interaction. Such difficulties arise in fact with local fields in nonlocal interaction.

D. Rivier (Lausanne).

Caianiello, E. R. Number of Feynman graphs and convergence. Nuovo Cimento (10) 3 (1956), 223-225.

Królikowski, Wojciech. The configurational equation of photons. Acta Phys. Polon. 14 (1955), 197-207. (Russian summary)

The Salpeter-Bethe equation for two photons is derived.

P. T. Matthews (Birmingham).

Kemmer, N.; and Salam, Abdus. On the relativistic equation for scattering. Proc. Roy. Soc. London. Ser. A. 230 (1955), 266-271.

An idea of Wick applied to a bound-state problem [Phys. Rev. (2) 96 (1954), 1124-1134; MR 16, 655] is here adapted to proton-neutron scattering. If M and m are nucleon and meson masses respectively, the gap $2M < E < 2M+m$ in the rest-mass spectrum allows the Bethe-Salpeter equation to be changed by analytical continuation into one with a kernel of elliptic type. The resulting integral equation involves the unknown function under both a four- and a three-dimensional integral. The authors sketch a possible numerical approach to the solution and in an appendix indicate that it could also be solved by Fredholm methods. They assert that the same method could be applied to meson-nucleon scattering for energies between $M+m$ and $M+2m$.

A. J. Coleman.

Gol'dman, I. I.; and Migdal, A. B. Theory of scattering in the quasiclassical approximation. Z. Eksper. Teoret. Fiz. 28 (1955), 394-400. (Russian)

The wave function is split into two parts, $\psi = \psi_0 + \psi_1$ where ψ_0 is the WKB approximation but ψ_1 is obtained by means of a Green's function which can be evaluated approximately if the trajectories of the corresponding classical problem are known. The method is illustrated by the case of scattering at high energies for which the trajectories are almost straight lines.

A. J. Coleman.

Humbert, J. Potentiels critiques et niveaux virtuels des noyaux atomiques. Bull. Soc. Roy. Sci. Liège 23, (1954), 148-163.

The present paper fills some gaps in a previous work of the author [Mém. Soc. Roy. Sci. Liège (4) 12 (1952), no. 4; MR 14, 171]. Consider the radial Schrödinger equation

$$(1) \quad \varphi'' + \left[k^2 - \frac{l(l+1)}{r^2} - v(r) \right] \varphi = 0,$$

where $\varphi(0)=0$. The values of k for which φ satisfies the condition

$$(2) \quad \lim_{r \rightarrow \infty} \left[\frac{\varphi'(r)}{\varphi(r)} - \frac{H_{l+1/2}^{(1)}(kr)}{H_{l+1/2}^{(1)}(kr)} \right] = 0$$

are called levels (real or virtual) of (1). If (2) is satisfied for $k=0$, then $v(r)$ is called a critical potential. The author shows that if $l \geq 1$ and $v(r)$ is a critical potential then $k=0$ is a double root of (2). Formulas are given to show how this double level splits under a small perturbation of $v(r)$ and the Breit-Wigner resonance formula for this case is obtained. Since $k=0$ is neither a zero nor a pole of the S-matrix, the author points out that the question of whether the S-matrix may have a double pole is still open.

B. Friedman (Berkeley, Calif.).

Collected papers on meson theory. I. Formalism and models. Progr. Theoret. Phys. Suppl. no. 1 (1955), 251 pp.

Following an introduction by S. Tomonaga, Progress in meson theory in Japan (pp. 1-6), there are reprinted by photo-offset 34 papers on the subject of the subtitle by various Japanese physicists.

Collected papers on meson theory. II. Intermediate and strong coupling theories. Progr. Theoret. Phys. Suppl. no. 2 (1955), 218 pp.

Photo-offset reprint of seven papers on the subject of the subtitle by Japanese physicists. Introduction (pp. i-ii) by H. Yukawa. The development of the method of approximation in meson theory in Japan.

Čavčanidze, V. V. On the interaction of boson-fermion fields. Dokl. Akad. Nauk SSSR (N.S.) 104 (1955), 205-208. (Russian)

Several sets of field equations for interacting fermion and boson fields are derived on the basis of various assumptions about the interaction.

N. Rosen (Haifa).

Prosperi, G. M.; e Tosi, C. Sulle connessioni matematiche fra le teorie classiche dell'elettrone di Feynman e di Rzewuski. Nuovo Cimento (10) 2 (1955), 1342-1344.

Thermodynamics, Statistical Mechanics

Popov, Kiril A. On a fundamental formula in the theory of irreversible thermodynamic processes. Dokl. Akad. Nauk SSSR (N.S.) 106 (1956), 422-424. (Russian)

Curtiss, C. F. Kinetic theory of nonspherical molecules. J. Chem. Phys. 24 (1956), 225-241.

Fierz, M. Der Ergodensatz in der Quantenmechanik. Helv. Phys. Acta 28 (1955), 705-715.

This present paper criticizes in two respects the standpoint adopted by von Neumann [Z. Physik 57 (1929), 30-70] and Pauli and Fierz [ibid. 106 (1937), 572-587] for the proof of the quantum-mechanical

ergodic theorem. This standpoint assumes an a priori probability distribution for the system of coarse-grained cells; it is then established that ergodicity prevails with overwhelming probability. The present paper argues that the choice of coarse-grained cells, being determined for actual systems by very definite considerations, cannot be treated as random. As a substitute the author proposes to treat as random the perturbing energies to which the approach to equilibrium can be ascribed. This idea is illustrated on an example but not developed in general. The author further criticizes the entropy definition adopted by von Neumann and argues that von Neumann's condition for ergodicity (non-vanishing of second energy differences) is too restrictive for relevant needs. The paper begins with a short discussion of the ergodic theorem and the entropy definition for classical systems.

L. Van Hove (Utrecht).

ter Haar, D. *Foundations of statistical mechanics.* Rev. Mod. Phys. 27 (1955), 289-338.

The author gives an excellent review of the extensive work carried out since the beginnings of statistical mechanics on the foundation problem of this branch of physics. The various lines of approach to the foundation problem, all of which were essentially initiated by Boltzmann, are considered in succession, for classical systems first, then for the quantum case. They are the H -theorem, the ergodic and quasi-ergodic theorems, and the use of representative ensembles for which an H -theorem can also be formulated. In the reviewer's opinion, the main merit of the author has been to give a clear analysis of the main developments, of their motivation and of their mutual relationships, without burdening the text with comments on too large a number of papers of secondary importance. This has enabled him to carry out the physical reasoning and the mathematical derivations with sufficient details to make the paper self-contained and readable without continuous reference to the original literature. Still, by quotations often mentioned between brackets, he gives a very complete survey of the literature, old and new, and the paper can be considered not only as a review of the present state of the foundation problem in statistical mechanics, but also as a very useful sketch of the past history of the subject.

For classical systems, the author treats in succession the H -theorem in kinetic theory, the paradoxes of Loschmidt and Zermelo and the statistical reformulation of the H -theorem which they made necessary. The statistical formulation of the H -theorem is briefly illustrated by means of the model proposed by Lorentz for the motion of electrons in a metal; the mathematical details of the model are given in the first appendix to the paper. The classical ergodic and quasi-ergodic hypotheses are then discussed, as well as the more recent work on the equivalence between time and ensemble averages, for systems with a metrically indecomposable energy surface, initiated by Birkhoff and von Neumann. The result of Fermi on the quasi-ergodicity of a class of "normal systems" is quoted and the proof is given in Appendix II. Birkhoff's ergodic theorem is proved in Appendix III. This part of the paper contains a statement to the effect that for any quasi-ergodic system the energy surface is metrically indecomposable. The author justifies it by saying that quasi-ergodicity implies the necessity for every orbit on the energy surface to pass through any region A of positive measure, a statement certainly valid for open A but not for a set A which is only measurable (as can be seen by

taking for A the set complementary to an orbit). Therefore as far as the reviewer can see and in conflict with the author's view, it is not established that quasi-ergodicity implies metric indecomposability. The analysis of classical systems ends with the treatment of representative ensembles.

A similar discussion is given for quantum statistics. For the kinetic form of the H -theorem an illustrative example is again considered (the Fermi-Dirac gas), with mathematical details worked out in appendix. Quantum ensemble theory is discussed, with mention of the various proposals which have been made in the course of time to explain the approach to equilibrium. The role of Klein's lemma is stressed and its proof is given in Appendix VI. The quantum-mechanical version of the ergodic theorem is finally treated, an interesting analysis which incorporates important remarks due to Fierz [see the paper reviewed above] and unpublished at the time of publication of the present paper. An appendix gives the proof. Detailed balancing is briefly commented upon in another appendix.

Both the classical and the quantum-mechanical sections of the review article end with summaries of the present situation. That these summaries cannot be considered completely conclusive nor completely satisfactory can of course not be blamed on the author. It is entirely due to the fact that the foundation problem of statistical mechanics is actually far from settled.

L. Van Hove (Utrecht).

Marquet, Simone. *Etude mathématique de l'équation de Boltzmann.* C. R. Acad. Sci. Paris 242 (1956), 615-617.

Ohne Beweis wird die Existenzaussage für Lösungen der Maxwell-Boltzmann'schen Gleichung (Modell der elastischen Kugeln) im sphärisch-symmetrischen, räumlich homogenen Fall der früheren Note [C. R. Acad. Sci. Paris 237 (1953), 1637-1640; MR 15, 435] verschärft und das Streben gegen die Maxwell'sche Gleichgewichtslösung behauptet.

D. Morgenstern (Berlin).

Teramoto, Ei; and Suzuki, Chieko. *The statistical mechanical aspect of H -theorem.* Progr. Theoret. Phys. 14 (1955), 411-422.

A simple one-dimensional model is considered for the approach to statistical equilibrium. It consists of one-dimensional perfectly hard particles with elastic reflection at the wall. The time evolution of the system for simple choices of the initial distribution can be calculated explicitly without assuming a persistent molecular chaos. The exact solution is compared with the solution of the corresponding transport equation valid under such a chaos hypothesis.

L. Van Hove (Utrecht).

Kaschlunn, F. *Zur Statistik eines Fermi-Dirac-Gases in Wechselwirkung mit einem Bose-Einstein-Gas.* Ann. Physik (6) 16 (1955), 257-286.

A Fermi gas in interaction with a Bose gas is considered in the lowest approximation, for which the hamiltonian can be written as an expression quadratic in the occupation numbers. A discussion is given of the grand canonical averages and mean square deviations of thermodynamical quantities and occupation numbers, taking into account the correlations between the latter. The case of very low temperature is considered in particular. The analysis is purely formal and no application is presented.

L. Van Hove (Utrecht).

See also: Bohm and Schützer, p. 752.

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